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Shifted matroid optimization

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ABSTRACT

We show that finding lexicographically minimal n bases in a matroid can be done in polynomial time in the oracle model. This follows from a more general result that the shifted optimization problem over a matroid can be solved in polynomial time as well. We also solve these problems for intersections of strongly base orderable matroids.

1. Introduction

Let G be a connected graph and let n be a positive integer. Given n spanning trees in G, an edge is *vulnerable* if it is used by all trees. We wish to find n spanning trees with minimum number of vulnerable edges. One motivation for this problem is as follows. We need to make a sensitive broadcast over G. In the planning stage, n trees are chosen and prepared. Then, just prior to the actual broadcast, one of these trees is randomly chosen and used. An adversary, trying to harm the broadcast and aware of the prepared trees but not of the tree finally chosen, will try to harm a vulnerable edge, used by all trees. So we protect each vulnerable edge with high cost, and our goal is to choose n spanning trees with minimum number of vulnerable edges.

Here we consider the following harder problem. For $k = 1, \ldots, n$, call an edge *k-vulnerable* if it is used by at least k of the n trees. We want to find n *lexicographically minimal* trees, that is, which first of all minimize the number of n-vulnerable edges, then of $(n-1)$-vulnerable edges, and so on. More precisely, given n trees, define their *vulnerability vector* to be $f = (f_1, \ldots, f_n)$ with f_k the number of k-vulnerable edges. Then, n trees with vulnerability vector f are better than n trees with vulnerability vector g if the last nonzero entry of $g - f$ is positive. (We remark that this order is often used in the symbolic computation literature, where it is called *reverse lexicographic*, but for brevity we will simply call it here *lexicographic*.) As a byproduct of our results, we show how to find in polynomial time n lexicographically minimal spanning trees, which in particular minimize the number of vulnerable edges.

This problem can be defined for any combinatorial optimization set as follows. For matrix $x \in \mathbb{R}^{d\times n}$, let x^j be its jth column. Define the *n-product of* a set $S \subseteq \mathbb{R}^d$ by

$$S^n := \times_n S := \{x \in \mathbb{R}^{d\times n} : x^j \in S,\ j = 1, \ldots, n\}.$$

Call two matrices $x, y \in \mathbb{R}^{d\times n}$ *equivalent* and write $x \sim y$ if each row of x is a permutation of the corresponding row of y. The *shift* of matrix $x \in \mathbb{R}^{d\times n}$ is the unique matrix $\bar{x} \in \mathbb{R}^{d\times n}$ which satisfies $\bar{x} \sim x$ and $\bar{x}^1 \geq \cdots \geq \bar{x}^n$, that is, the unique matrix equivalent to x with each row nonincreasing. Let $|x^j| := \sum_{i=1}^d |x_{i,j}|$ and $|x| := \sum_{j=1}^n |x^j|$ be the sums of absolute values of the components of x^j and x, respectively. The *vulnerability vector* of $x \in \{0, 1\}^{d\times n}$ is

$$(|\bar{x}^1|, \ldots, |\bar{x}^n|).$$

We then have the following nonlinear combinatorial optimization problem.

Lexicographic combinatorial optimization. Given $S \subseteq \{0, 1\}^d$ and n, solve

$$\text{lexmin}\{(|\bar{x}^1|, \ldots, |\bar{x}^n|) : x \in S^n\}. \quad (1)$$

The complexity of this problem depends on the presentation of the set S. In [7], it was shown that it is polynomial time solvable for $S = \{z \in \{0, 1\}^d\ :\ Az = b\}$ for any totally unimodular A and any integer b. Here we solve the problem for matroids.

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Theorem 1.1. *The lexicographic combinatorial optimization problem (1) over the bases of any matroid given by an independence oracle and any n is polynomial time solvable.*

Our spanning tree problem is the special case with S the set of indicators of spanning trees in a given connected graph with d edges, and hence is polynomial time solvable.

We note that Theorem 1.1 provides a solution of a nonlinear optimization problem over matroids, adding to available solutions of other nonlinear optimization problems over matroids and independence systems in the literature, see e.g. [2,3,8,9] and the references therein.

We proceed as follows. In Section 2, we discuss the *shifted combinatorial optimization problem* and its relation to the lexicographic combinatorial optimization problem. In Section 3, we solve the shifted problem over matroids in Theorem 3.4 and conclude Theorem 1.1. In Section 4 we discuss matroid intersections, partially solve the shifted problem over the intersection of two strongly base orderable matroids (which include gammoids) in Theorem 4.3, and leave open the complexity of the problem for the intersection of two arbitrary matroids. We conclude in Section 5 with some final remarks about the polynomial time solvability of the shifted and lexicographic problems over totally unimodular systems from [7], and show that these problems are *NP*-hard over matchings already for $n = 2$ and cubic graphs.

2. Shifted combinatorial optimization

Lexicographic combinatorial optimization can be reduced to the following problem.

Shifted combinatorial optimization. Given $S \subseteq \{0, 1\}^d$ and $c \in \mathbb{Z}^{d\times n}$, solve

$$\max\{\bar{c}\,\bar{x} : x \in S^n\}. \tag{2}$$

The following lemma was shown in [7]. We include the proof for completeness.

Lemma 2.1 (*[7]*). *The Lexicographic Combinatorial Optimization problem (1) can be reduced in polynomial time to the Shifted Combinatorial Optimization problem (2).*

Proof. Define the following $c \in \mathbb{Z}^{d\times n}$, and note that it satisfies $\bar{c} = c$,

$$c_{i,j} := -(d+1)^{j-1}, \quad i = 1, \ldots, d,\ j = 1, \ldots, n.$$

Consider any two vectors $x, y \in S^n$, and suppose that the vulnerability vector $(|\bar{x}^1|, \ldots, |\bar{x}^n|)$ of x is lexicographically smaller than the vulnerability vector $(|\bar{y}^1|, \ldots, |\bar{y}^n|)$ of y. Let r be the largest index such that $|\bar{x}^r| \neq |\bar{y}^r|$. Then, $|\bar{y}^r| \geq |\bar{x}^r| + 1$. We then have

$$\begin{aligned}\bar{c}\,\bar{x} - \bar{c}\,\bar{y} = c\,\bar{x} - c\,\bar{y} &= \sum_{j=1}^{n}(d+1)^{j-1}\left(|\bar{y}^j| - |\bar{x}^j|\right) \\ &\geq \sum_{j<r}(d+1)^{j-1}\left(|\bar{y}^j| - |\bar{x}^j|\right) + (d+1)^{r-1} \\ &\geq (d+1)^{r-1} - \sum_{j<r} d(d+1)^{j-1} > 0.\end{aligned}$$

Thus, an optimal solution x for problem (2) is also optimal for problem (1). □

We proceed to reduce the Shifted Combinatorial Optimization problem (2) in turn to two yet simpler auxiliary problems. For a set of matrices $U \subseteq \{0, 1\}^{d\times n}$ let $[U]$ be the set of matrices which are equivalent to some matrix in U,

$$[U] := \{x \in \{0, 1\}^{d\times n} : \exists\, y \in U,\ x \sim y\}.$$

Consider the following two further algorithmic problems over a given $S \subseteq \{0, 1\}^d$:

Shuffling. Given $c \in \mathbb{Z}^{d\times n}$, solve $\max\{cx : x \in [S^n]\}$. (3)

Fiber. Given $x \in [S^n]$, find $y \in S^n$ such that $x \sim y$. (4)

Lemma 2.2. *The Shifted Combinatorial Optimization problem (2) can be reduced in polynomial time to the Shuffling and Fiber problems (3) and (4).*

Proof. First solve the Shuffling problem (3) with profit matrix $\bar{c}$ and let $x \in [S^n]$ be an optimal solution. Next solve the Fiber problem (4) for x and find $y \in S^n$ such that $x \sim y$. We claim that y is optimal for the Shifted Combinatorial Optimization problem (2). To prove this, we consider any z which is feasible in (2), and prove that the following inequality holds,

$$\bar{c}\,\bar{y} = \bar{c}\,\bar{x} \geq \bar{c}\,x \geq \bar{c}\,\bar{z}.$$

Indeed, the first equality follows since $x \sim y$ and therefore we have $\bar{y} = \bar{x}$. The middle inequality follows since $\bar{c}$ is nonincreasing. The last inequality follows since $z \in S^n$ implies that $\bar{z} \in [S^n]$ and hence $\bar{z}$ is feasible in (3). So y is indeed an optimal solution for problem (2). □

3. Matroids

Let E be any finite set, in particular $E = [d] := \{1, \ldots, d\}$ or $E = [d] \times [n]$. We make the following definitions. The *n-union* of a set $S \subseteq \{0, 1\}^E$ is defined to be the set

$$\vee_n S := \left\{x \in \{0, 1\}^E : \exists x_1, \ldots, x_n \in S,\ x = \sum_{k=1}^{n} x_k\right\}.$$

We call a set $S \subseteq \{0, 1\}^E$ a *matroid* if it is the set of indicators of independent sets of a matroid over E. The following facts about n-unions of matroids are well known, see e.g. [10].

Proposition 3.1. *For any matroid S and any n we have that $\vee_n S$ is also a matroid. Given an independence oracle for S, it is possible in polynomial time to realize an independence oracle for $\vee_n S$, and if $x \in \vee_n S$, to find $x_1, \ldots, x_n \in S$ with $x = \sum_{k=1}^n x_k$.*

Define the *n-lift* of a set of vectors $S \subseteq \{0, 1\}^d$ to be the following set of matrices,

$$\uparrow_n S := \left\{x \in \{0, 1\}^{d\times n} : \sum_{j=1}^{n} x^j \in S\right\}.$$

We need two lemmas.

Lemma 3.2. *For any set $S \subseteq \{0, 1\}^d$ and any n we have the equality $[S^n] = \vee_n \uparrow_n S$.*

Proof. Consider $x \in [S^n]$. Then $x \sim y$ for some $y \in S^n$, and thus $y^j \in S$ for $j = 1, \ldots, n$. Since $x \sim y$, each row of x is a permutation of the corresponding row of y. Assume that the ith row of x is given by the permutation π_i of the corresponding row of y. That is, $x_{i,j} = y_{i,\pi_i(j)}$. For $k = 1, \ldots, n$, we define a matrix $z_k \in \{0, 1\}^{d\times n}$ whose column sum satisfies $\sum_{j=1}^n z_k^j = y^k$, by $(z_k)_{i,j} := x_{i,j}$ if $\pi_i(j) = k$, and otherwise $(z_k)_{i,j} := 0$. Since $y^k \in S$, we conclude that $z_k \in \uparrow_n S$. Since the supports of the z_k are pairwise disjoint, we have that $\sum_{k=1}^n z_k \in \vee_n \uparrow_n S$. However, $\sum_{k=1}^n z_k = x$ by definition, and thus $x \in \vee_n \uparrow_n S$.

In the other direction, assume that $x \in \vee_n \uparrow_n S$. Then, there are $x_1, \ldots, x_n \in \uparrow_n S$ such that $x = \sum_{k=1}^n x_k$. That is, there are $x_1, \ldots, x_n \in \{0, 1\}^{d\times n}$ such that for each $k = 1, \ldots, n$, we have