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Uniformly monotone functions — Definition, properties, characterizations

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ABSTRACT

Quasi-concave functions play an important role in economics and finance as utility functions, measures of risk or the other objects used, mainly, in portfolio selection analysis. Unfortunately, their limited application is due to the fact that their supremum, sum, product are typically not quasi-concave. This difficulty is overcome by establishing uniformly quasi-concave functions. We contribute with a new characterization of uniformly quasi-concave functions that allows easier verification and provide more straightforward insight.

1. Introduction

Let us open our discussion with a definition of a quasi-concave real function.

Definition 1. Let $E \subset \mathbb{R}^n$. We say that $f : E \to \mathbb{R}$ is quasi-concave if

1. E is convex.
2. For each $\Delta \in \mathbb{R}$ the level set $\mathrm{lev}_{\geq \Delta} f = \{x \in E : f(x) \geq \Delta\}$ is convex.

Alternatively, one can deal with quasi-convex functions; i.e. $-f$ is quasi-concave. Quasi-concave functions are useful in economics and finance; see [1–4,8,7]. An application of quasi-concave functions is limited, because typically supremum, sum, product of quasi-concave functions are not quasi-concave. This difficulty is overcome by establishing uniformly quasi-concave functions, due to Prékopa et al. [6].

Definition 2. Let $E \subset \mathbb{R}^n$ and $m \in \mathbb{N}$. We say functions $f_i : E \to \mathbb{R}$, $i = 1, 2, \ldots, m$ are uniformly quasi-concave if

1. E is convex.
2. For each $i = 1, 2, \ldots, m$ the function f_i is quasi-concave.
3. For each $x, y \in E, x \neq y$ either

   $\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(x)$

   or

   $\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(y).$

We present a generalization and equivalent descriptions of uniformly quasi-concave functions. This paper improves our considerations published in [5].

2. Introductory definitions

We will deal with functions defined on a common nonempty set E and having values in a totally ordered set $\mathcal{Q} = (\mathcal{Q}, \prec_{\mathcal{Q}})$, e.g. $\mathcal{Q}$ could be the extended real line $\mathbb{R}^* = [-\infty, +\infty]$ equipped with the natural ordering. On $\mathcal{Q}$, we will employ derived relations $\succ_{\mathcal{Q}}, \preceq_{\mathcal{Q}}, \succeq_{\mathcal{Q}}, \min_{\mathcal{Q}}, \max_{\mathcal{Q}}$. The set of all such functions will be denoted by $\mathcal{F}(E, \mathcal{Q})$.

Let us start with definitions of main objects of our interest.

Definition 3. We say a nonempty family $\mathscr{F} \subset \mathcal{F}(E, \mathcal{Q})$ is uniformly monotone if for each $x, y \in E, x \neq y$ either

$$\forall f \in \mathscr{F} \quad \min_{\mathcal{Q}}\{f(x), f(y)\} = f(x)$$

or

$$\forall f \in \mathscr{F} \quad \min_{\mathcal{Q}}\{f(x), f(y)\} = f(y).$$

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We are searching for equivalent descriptions of uniformly monotone families of functions. We will use level sets for that. Let us recall appropriate definitions.

Definition 4. For a function $f : E \to \mathcal{Q}$ and a given level $\Delta \in \mathcal{Q}$ we consider a level set

$$\mathrm{lev}_{\pitchfork\Delta} f = \{x \in E : f(x) \pitchfork \Delta\},$$

and, for a nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$, we consider a set of all its level sets

$$\mathrm{LEVELs}_{\pitchfork} \mathcal{F} = \{\mathrm{lev}_{\pitchfork\Delta} f : f \in \mathcal{F}, \ \Delta \in \mathcal{Q}\},$$

where $\pitchfork$ states for a proper relation sign $\preceq_{\mathcal{Q}}, \prec_{\mathcal{Q}}, \succeq_{\mathcal{Q}}, \succ_{\mathcal{Q}}$.

Working in a vector space $\mathcal{V}$, convex sets are well-defined and we can correctly define quasi-concave functions.

Definition 5. Let $\mathcal{V}$ be a vector space and $E \subset \mathcal{V}$ be nonempty. We say that $f \in \mathcal{F}(E, \mathcal{Q})$ is quasi-concave if

- E is convex.
- For each $\Delta \in \mathcal{Q}$ the level set $\mathrm{lev}_{\succeq_{\mathcal{Q}} \Delta} f$ is convex.

The definition of uniformly quasi-concave functions introduced in [6] can be also generalized for our setting.

Definition 6. Let $\mathcal{V}$ be a vector space and $E \subset \mathcal{V}$ be nonempty. We say that a nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ is uniformly quasi-concave if

- E is convex,
- each function of $\mathcal{F}$ is quasi-concave,
- $\mathcal{F}$ is uniformly monotone.

3. Descriptions of uniformly monotone functions

3.1. Partial ordering induced by functions

Any set of functions determines a partial ordering on their common domain.

Definition 7. A nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ determines a partial ordering $\prec^{\mathcal{F}}$ and an equivalence $\sim^{\mathcal{F}}$ on E by

$$x \prec^{\mathcal{F}} y \iff \begin{array}{l} \forall f \in \mathcal{F} : f(x) \preceq_{\mathcal{Q}} f(y), \\ \exists g \in \mathcal{F} \quad \text{s.t. } g(x) \prec_{\mathcal{Q}} g(y), \end{array}$$

$$x \sim^{\mathcal{F}} y \iff \forall f \in \mathcal{F} : f(x) = f(y).$$

An equivalent characterization of uniformly monotone families of functions is based on this induced partial ordering.

Theorem 8. *Let $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ be a nonempty family. Then $\mathcal{F}$ is uniformly monotone iff the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$, i.e. for each couple $x, y \in E$ just one from three following relations holds*

$$x \prec^{\mathcal{F}} y, \qquad x \sim^{\mathcal{F}} y, \qquad y \prec^{\mathcal{F}} x.$$

Proof. We are proving an equivalence.

1. Let $\mathcal{F}$ be uniformly monotone.
   Fix $x, y \in E$, $x \nsim^{\mathcal{F}} y$.
   Then, there is a function $g \in \mathcal{F}$ s.t. $g(x) \neq g(y)$.
   We have to distinguish two possibilities:
   (a) Let $g(x) \prec_{\mathcal{Q}} g(y)$.
   Hence from uniform monotonicity, $\forall f \in \mathcal{F} : f(x) \preceq_{\mathcal{Q}} f(y)$.
   Consequently, $x \prec^{\mathcal{F}} y$.
   (b) Let $g(x) \succ_{\mathcal{Q}} g(y)$.
   Hence from uniform monotonicity, $\forall f \in \mathcal{F} : f(x) \succeq_{\mathcal{Q}} f(y)$.
   Consequently, $y \prec^{\mathcal{F}} x$.
   We have proved the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$.
2. Let the factor space $E/_{\sim^{\mathcal{F}}}$ be totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$.
   Fix $x, y \in E$.
   Since the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$, we have to distinguish three possibilities:
   (a) If $x \sim^{\mathcal{F}} y$ then $\forall f \in \mathcal{F} : f(x) = f(y)$.
   Hence, $\forall f \in \mathcal{F} : \min_{\mathcal{Q}}\{f(x), f(y)\} = f(x) = f(y)$.
   (b) If $x \prec^{\mathcal{F}} y$ then $\forall f \in \mathcal{F} : f(x) \preceq_{\mathcal{Q}} f(y)$.
   Hence, $\forall f \in \mathcal{F} : \min_{\mathcal{Q}}\{f(x), f(y)\} = f(x)$.
   (c) If $y \prec^{\mathcal{F}} x$ then $\forall f \in \mathcal{F} : f(x) \succeq_{\mathcal{Q}} f(y)$.
   Hence, $\forall f \in \mathcal{F} : \min_{\mathcal{Q}}\{f(x), f(y)\} = f(y)$.
   We have shown $\mathcal{F}$ is uniformly monotone.

3.2. The set of all level sets

Another equivalent characterization can be received using level sets.

Theorem 9. *Let $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ be a nonempty family. Then the following are equivalent:*

1. *$\mathcal{F}$ is uniformly monotone.*
2. *$\mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$ is totally ordered by natural set-ordering.*
3. *$\mathrm{LEVELs}_{\prec_{\mathcal{Q}}} \mathcal{F}$ is totally ordered by natural set-ordering.*
4. *$\mathrm{LEVELs}_{\succeq_{\mathcal{Q}}} \mathcal{F}$ is totally ordered by natural set-ordering.*
5. *$\mathrm{LEVELs}_{\succ_{\mathcal{Q}}} \mathcal{F}$ is totally ordered by natural set-ordering.*

Proof. It is sufficient to prove the equivalence for $\mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$, since a type of level sets is totally ordered if and only if the other types of level sets are totally ordered. That is because $\mathrm{LEVELs}_{\succ_{\mathcal{Q}}} \mathcal{F} = E \setminus \mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$ and $\mathrm{LEVELs}_{\prec_{\mathcal{Q}}} \mathcal{F} = \mathrm{LEVELs}_{\succ^{\pitchfork}_{\mathcal{Q}}} \mathcal{F}$, $\mathrm{LEVELs}_{\succeq_{\mathcal{Q}}} \mathcal{F} = \mathrm{LEVELs}_{\preceq^{\pitchfork}_{\mathcal{Q}}} \mathcal{F}$, where $\prec^{\pitchfork}_{\mathcal{Q}}$ denotes the reverse ordering to $\prec_{\mathcal{Q}}$.

1. Let $\mathcal{F}$ be uniformly monotone.
   Let $A, B \in \mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$ and $A \setminus B \neq \emptyset$.
   Then, $A = \mathrm{lev}_{\preceq_{\mathcal{Q}} \alpha} f$, $B = \mathrm{lev}_{\preceq_{\mathcal{Q}} \beta} g$ for some $f, g \in \mathcal{F}$ and $\alpha, \beta \in \mathcal{Q}$.
   Moreover, there is $\xi \in E$ such that $\xi \in A$ and $\xi \notin B$, i.e. $f(\xi) \preceq_{\mathcal{Q}} \alpha$ and $g(\xi) \succ_{\mathcal{Q}} \beta$.
   Take $x \in B$; i.e. $g(x) \preceq_{\mathcal{Q}} \beta$.
   Then, $g(x) \prec_{\mathcal{Q}} g(\xi)$. Accordingly to uniform monotonicity, $f(x) \preceq_{\mathcal{Q}} f(\xi)$.
   Thus, $f(x) \preceq_{\mathcal{Q}} \alpha$ and, consequently, $x \in A$.
   We have derived $A \supset B$. That means that $\mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$ is totally ordered by set-ordering.
2. Let $\mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$ be totally ordered by set-ordering.
   Take $x, y \in E$.
   Assume $f, g \in \mathcal{F}$ such that $f(x) \prec_{\mathcal{Q}} f(y)$ and $g(x) \succ_{\mathcal{Q}} g(y)$.
   Denoting $\alpha = f(x)$, $\beta = g(y)$, we observe
   $x \in \mathrm{lev}_{\preceq_{\mathcal{Q}} \alpha} f$, $y \notin \mathrm{lev}_{\preceq_{\mathcal{Q}} \alpha} f$,
   $x \notin \mathrm{lev}_{\preceq_{\mathcal{Q}} \beta} g$, $y \in \mathrm{lev}_{\preceq_{\mathcal{Q}} \beta} g$.
   Therefore, $\mathrm{lev}_{\preceq_{\mathcal{Q}} \alpha} f \neq \mathrm{lev}_{\preceq_{\mathcal{Q}} \beta} g$, $\mathrm{lev}_{\preceq_{\mathcal{Q}} \alpha} f \not\subset \mathrm{lev}_{\preceq_{\mathcal{Q}} \beta} g$, $\mathrm{lev}_{\preceq_{\mathcal{Q}} \alpha} f \not\supset \mathrm{lev}_{\preceq_{\mathcal{Q}} \beta} g$, which is in contradiction with the assumption that $\mathrm{LEVELs}_{\preceq_{\mathcal{Q}}} \mathcal{F}$ is totally ordered by set-ordering.
   We derived the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$.
   Accordingly to Theorem 8, we have shown $\mathcal{F}$ is uniformly monotone.

3.3. Composition of functions

Derived characterization by means of level sets implies a characterization using composition of appropriate functions.