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Optimal shift coloring of trees

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ABSTRACT

This paper was motivated by the problem of scheduling the openings of pharmacies during week-ends and holiday periods (shifts). The problem can be modeled as a coloring problem on a graph. In this paper we focus on the special case where the underlying graph is a tree, or, more generally, it is endowed with a tree-metric, and we provide a polynomial-time algorithm. We also provide direct optimal solutions for special trees like stars and paths.

1. Introduction

This paper was motivated by the problem of scheduling the openings of pharmacies during week-ends and holiday periods (see [1,3]). In practice, each pharmacy has to be assigned to one of a given number K of shifts, meaning that each pharmacy will remain open 24 h a day during one among K consecutive week-ends. In other words, one has to partition the set of all pharmacies into K subsets, each corresponding to a particular shift. This may be viewed as a coloring problem with different colors corresponding to different shifts. In fact, it is convenient to look at the set of pharmacies as the set V of vertices of an undirected graph $G(V, E)$. We also assume that the graph G is connected and for every edge $\{i, j\} \in E$ a positive length $l(i, j)$ is given. Each edge $\{i, j\}$ may be thought as a direct connection (e.g. a road) between vertices (pharmacies) i and j, and the length $l(i, j)$ is either the length of this connection or the time required to go over it. We also denote by d_{ij} the shortest distance between i and j, according to E and l.

In the paper, we assume that the number K of shifts is given and without loss of generality $2 \le K \le |V| - 1$. We let $[K]$ be the set of colors $\{1, 2 \dots, K\}$. We define *K-Shift Coloring* as any coloring χ of all vertices in V using all K colors. Denote by $\chi(i)$ the color of the vertex i and let $V_1, V_2, \dots, V_K$ be the subsets of V having colors $1, 2, \dots, K$.

We are interested in finding an optimal K-Shift Coloring χ, i.e., a K-Shift Coloring such that the total distance

$$D(\chi) = \sum_{i\in V} \sum_{k\in[K]} \min_{j\in V_k} d_{ij}$$

is minimized.

Note that any K-Shift Coloring identifies a partition of V into K subsets.

To our knowledge, the only works devoted to this particular graph-vertex coloring problem are [1–3]. In particular, in [2] it is shown that, in general, the problem is NP-hard for $K \ge 3$ (by reduction from the domatic number problem) and polynomial for $K = 2$. In this paper we consider the simplest version of such a problem where we assume that the underlying graph G is a tree, or, more generally, G is endowed with a tree-metric (see [4] for definition and important properties of tree-metrics). We will show that, in this case, the problem is polynomial. After introducing the notion of *Perfect K-Shift Coloring* and presenting a sufficient optimality condition that holds for arbitrary graphs (Section 2), in Section 3 we directly provide the optimal K-Shift Coloring of special trees like stars and paths. In Section 4, we describe the algorithm for finding an optimal K-Shift Coloring of a tree with unit distances and, in Section 5, we do the same for trees with arbitrary (positive) distances.

2. Perfect *K*-Shift Coloring of graphs

Given a generic graph $G = (V, E)$, a vertex v and a set of K vertices $Q \subseteq V$, let us denote by $D(v, Q)$ the sum of the distances from vertex v to each of the K vertices in Q. Call *K-Core Neighborhood of*

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v, and denote it by $\mathcal{C}_v$, any set of K vertices in G closest to v (including vertex v itself), i.e., $D(v, \mathcal{C}_v) = \min\{D(v, Q) : |Q| = K, Q \subseteq V\}$. Let us denote by $d^*(v)$ this minimum, i.e., $d^*(v) = D(v, \mathcal{C}_v)$. A generic K-Shift Coloring is *full with respect to* v if there exists a K-Core Neighborhood of v such that all its vertices have different colors. Let us call *Full* such a K-Core Neighborhood of v and denote it by $\mathcal{F}_v$. Notice that, for any given v, $\mathcal{F}_v$ may or may not exist, and may or may not be unique. If $\mathcal{F}_v$ exists, then $D(v, \mathcal{F}_v) = d^*(v)$.

We remark that the notion of K-Core Neighborhood is independent of any coloring, while that of Full K-Core Neighborhood depends on the selected coloring χ.

Given a graph G, let us call *Perfect K-Shift Coloring* of G a coloring $\chi^\sharp$ such that $\chi^\sharp$ is full with respect to every vertex v, i.e., there exists a Full K-Core Neighborhood of v, for any vertex v.

Notice that a Perfect K-Shift Coloring of G may not exist. The circuit with 4 vertices and unit distances is a simple example of a graph for which there is no Perfect 3-Shift Coloring. We will show in Sections 4 and 5 that every tree possesses a Perfect K-Shift Coloring.

The following theorem provides an optimality sufficient condition, holding for any graph G (and not just for trees) and for any type of distances.

Theorem 1. *Any Perfect K-Shift Coloring χ^* of a graph G is also an optimal K-Shift Coloring of G.*

Proof. Given any K-Shift Coloring χ of G, let $V_1, V_2, \ldots, V_K$ and $D(\chi)$ be defined as in Section 1. We have: $D(\chi) = \sum_{v\in V}\sum_{k\in[K]} \min_{j\in V_k} d_{vj} \geq \sum_{v\in V} D(v, \mathcal{C}_v) = \sum_{v\in V} d^*(v)$ and $D(\chi^*) = \sum_{v\in V} D(v, \mathcal{F}_v) = \sum_{v\in V} D(v, \mathcal{C}_v) = \sum_{v\in V} d^*(v)$. This proves that $D(\chi) \geq D(\chi^*)$. □

Furthermore, the proof of Theorem 1 also shows the following result.

Theorem 2. *If a graph G possesses a Perfect K-Shift Coloring, then all optimal K-Shift Coloring of G are perfect.*

In the following sections we will show that every tree possesses a Perfect K-Shift Coloring, and how to build one.

3. *K*-Shift Coloring of special trees

Let us consider the case where the graph is a tree and, as a special case, assume that T is a star with unit distances. We have the following proposition.

Proposition 1. *Let T be a star with unit distances and vertices numbered $1, 2, \ldots, n$, where vertex 1 is the center. Any K-Shift Coloring such that vertex 1 gets color 1 and every pendant vertex gets a color between 2 and K, with at least one vertex per color, is a Perfect K-Shift Coloring of T and thus it is optimal and, vice-versa, every optimal K-Shift Coloring of T is of this form (up to a permutation of colors).*

Proof. Although the proposition is a simple corollary of Theorems 1 and 2, we provide here an alternative direct proof. Let χ be any K-Shift Coloring of T. We may assume, without loss of generality that $\chi(1) = 1$. Let $m_1, m_2, \ldots, m_K$ be the number of pendant vertices of T that are given colors $1, 2, \ldots, K$ respectively. Notice that $m_1 + m_2 + \cdots + m_K = n - 1$. Of course $m_j \geq 1$. The total distance that has to be traveled during shift 1 is

$$D_1 = 0\cdot(1+m_1) + 1\cdot\sum_{k>1} m_k = n-1-m_1$$

and during shift h $(2 \leq h \leq K)$

$$D_h = 1 + 2\cdot\sum_{k>1,k\neq h} m_k = 1 + 2\,(n-1-m_h)$$

so that the total distance is

$$\begin{aligned} &n-1-m_1+\sum_{h>1}(1+2\,(n-1-m_h)) \\ &= n-1-m_1+K-1+2\,(n-1)\,(K-1)-2\,(n-1-m_1) \\ &= m_1-3\,n+(2\,n-1)\,K+2. \end{aligned}$$

Evidently the total distance is minimized if and only if $m_1 = 0$, and its optimal value is equal to

$$(2\,n-1)\,(K-1)-(n-1)$$

no matter what the values $m_2, \ldots m_K$ are, provided that they are all at least 1. This proves that any optimal K-Shift Coloring has the form described in the statement of the theorem. It is also evident that such a K-Shift Coloring is perfect. □

The hypothesis of unit distances can actually be relaxed, as stated in the following proposition.

Proposition 2. *If T is a star with arbitrary distances, and the vertices are numbered from 1 to n according to their increasing distance from the center (vertex 1), then the K-Shift Coloring χ^* defined as*

$$\chi^*(v) = \min(v, K) \quad v = 1, \ldots, n,$$

is perfect and thus optimal. Furthermore, every optimal K-Shift Coloring is of this form (up to a permutation of colors).

Proof. If $v \leq K$ then the subset of vertices $\{1, 2, \ldots, K\}$ is $\mathcal{C}_v$ that is also $\mathcal{F}_v$. If $v > K$ then the subset of vertices $\{1, 2, \ldots, K-1, v\}$ is $\mathcal{C}_v$, that, again, is also $\mathcal{F}_v$. Hence, the given K-Shift Coloring is perfect and, by Theorem 1, optimal. It is also evident that any Perfect K-Shift Coloring must have this form (up to a permutation of colors). By Theorem 2 it follows that all optimal K-Shift Coloring must have this form (up to a permutation of colors). □

As a corollary of Theorem 1, we also have the following proposition.

Proposition 3. *If T is a path with arbitrary distances and its vertices are numbered $1, 2, \ldots, n$ in a canonical way (i.e., vertex 1 is a pendant vertex and the number of edges in the path from vertex 1 to vertex j is $j-1$, for any $j = 2, \ldots, n$), then a Perfect (and optimal) K-Shift Coloring of T is $\chi^*(j) = j \bmod K$.*

One may wonder if every Perfect K-Shift Coloring of a graph has the form described in Proposition 3. The answer is no, as the following example shows. Let $K = 2$ and let P be the path with 4 vertices and 3 edges where $l(1, 2) = l(3, 4) = 1$ and $l(2, 3) = 2$. For this path, the following K-Shift Coloring: $\chi^\sharp(1) = \chi^\sharp(4) = 1$ and $\chi^\sharp(2) = \chi^\sharp(3) = 2$ is also perfect (and thus optimal), but it is not of the form described in Proposition 3.

4. *K*-Shift Coloring of trees with unit distances

The previous Propositions 2 and 3 seem to indicate that the order in which vertices are colored is important. This is confirmed in the general case by the following algorithm. Given a tree T with unit distances, an optimal K-Shift Coloring of T may be found in polynomial time by the following algorithm.

Algorithm 1. Given a tree T and a number of colors K, do:

Step 1: **Ordering of vertices and initial colored subtree.** Choose any vertex of T as root (denote it by r) and order the vertices from 1 to n according to a breadth first exploration. Color the first K vertices with the K different colors, obtaining a colored subtree T_K. Set $m = K$.

Step 2: **Inductive construction of colored subtrees.** Consider vertex $w := m + 1$ and let $T_{m+1} = T_m \cup \{w\}$ be the subtree containing the vertices from 1 to $m + 1$. For each $k \in$