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Multi-objective optimisation of positively homogeneous functions and an application in radiation therapy

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ABSTRACT

Unconstrained multi-objective optimisation problems with p positively homogeneous objective functions are considered. We prove that such problems reduce to multi-objective optimisation problems with $p-1$ objectives and a single equality constraint. Thus, problems with two objectives can be solved with standard single objective optimisation methods and, for problems with $p>2$ objectives, we can compute infinitely many efficient solutions by solving a finite number of single objective problems. The proposed procedure is applied on radiotherapy for cancer treatment.

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1. Introduction

Decision making under a set of conflicting objectives is a common scenario in real life. Usually there exists a trade-off between objectives such that an ideal solution, one in which all objectives are optimised simultaneously, does not exist. Instead, we seek a set of solutions in which no objective can be improved without impairing at least one of the other objectives. These solutions are known as efficient solutions. Therefore, solving an optimisation problem considering multiple objectives requires finding all efficient solutions or finding the entire set of non-dominated points (the images of the efficient solutions) in outcome space. In this paper, we adopt the latter interpretation. For a comprehensive exposition of multi-objective optimisation, see [3].

During the last decades, substantial efforts have been made to develop strategies to solve multi-objective optimisation problems. In this paper we address such problems in which all the objectives are positively homogeneous (PH) functions, where a function $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ is positively homogeneous if $f(\lambda x)=\lambda f(x)$ for all $\lambda>0$ and all $x\in\mathbb{R}^n$. For convenience we also assume that $f(0)=0$.

In this section we first define the problem that is addressed in this paper, which we term the positively homogeneous multi-objective optimisation problem (PHMOP). Then we reduce the PHMOP with p objectives to a PHMOP with $(p-1)$ objective functions. The special case when $p=2$ is highlighted. Finally, we present a procedure that allows us to solve the original p-objective PHMOP by solving the reduced $(p-1)$-objective PHMOP.

The PHMOP can be stated as follows:

$$\text{PHMOP:}\ \underset{x\in\mathbb{R}^n}{\text{minimise}}\ f(x),$$

where $x\in\mathbb{R}^n$ is a vector of n decision variables x_i, $i=1,2,\ldots,n$, and f is a vector of p objective functions f_k, $k=1,2,\ldots,p$, each of which is positively homogeneous. To ensure PHMOP is well posed, we require that there is some x, k and l with $f_k(x)$ and $f_l(x)$ having opposite signs.

Throughout the paper we will use the following notation for the comparison of vectors. Let $y^1, y^2\in\mathbb{R}^p$. We write $y^1 \leqq y^2$ if $y^1_k \leqq y^2_k$ for all $k=1,\ldots,p$; $y^1 \leq y^2$ if $y^1 \leqq y^2$ but $y^1 \neq y^2$; and $y^1 < y^2$ if $y^1_k < y^2_k$ for all $k=1,\ldots,p$. We say that $y^1\in\mathbb{R}^p$ *dominates* $y^2\in\mathbb{R}^p$ if $y^1 \leq y^2$. A solution $\hat{x}\in\mathbb{R}^n$ is called an *efficient* solution of PHMOP if there is no $x\in\mathbb{R}^n$ where $f(x)$ dominates $f(\hat{x})$. In this case $f(\hat{x})$ is called a *non-dominated*point of PHMOP. We let X_E denote the set of all efficient solutions of PHMOP and denote the set of all non-dominated points as $Y_N := f(X_E)$. Clearly for any efficient solution

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$\hat{x} \in X_E$ of PHMOP with $f(\hat{x}) \neq 0$ we must have $f_l(\hat{x}) < 0 < f_k(\hat{x})$ for some k and l.

Proposition 1. *If $\hat{x}$ is an efficient solution of problem PHMOP, then $\lambda\hat{x}$ is an efficient solution of PHMOP for all $\lambda > 0$.*

Proof. Since f is a vector of positively homogeneous functions we have $f(\lambda\hat{x}) = \lambda f(\hat{x})$ for all $\lambda > 0$. Now, suppose that there exists a vector $x \in \mathbb{R}^n$ such that $f(x) \leq f(\lambda\hat{x})$. Then

$$f\left(\frac{1}{\lambda}x\right) = \frac{1}{\lambda}f(x) \leq \frac{1}{\lambda}f(\lambda\hat{x}) = f(\hat{x}),$$

a contradiction. □

It follows from Proposition 1 that we can obtain an infinite number of non-dominated points of PHMOP for each non-dominated point found when solving a new problem in which one objective has its value fixed. For ease of notation, we assume that the p'th objective is fixed (with the objectives being renumbered, if required), giving a new problem PHMOP$_p(t)$ with a constraint $f_p(x) = t$ for some (feasible) $t \in \mathbb{R}$ in which we optimise over the remaining $p-1$ objective functions f_k, $k = 1, 2, \ldots, p-1$. By t being feasible we mean that there is some $x \in \mathbb{R}^n$ such that $f_p(x) = t$. To formally define PHMOP$_p(t)$ we denote its vector of objective functions as $f'(x) = (f_1(x), f_2(x), \ldots, f_{p-1}(x))$, giving:

$$\begin{aligned} &\text{PHMOP}_p(t): \\ &\quad \underset{x\in\mathbb{R}^n}{\text{minimise}}\ f'(x) \\ &\quad \text{subject to } f_p(x) = t. \end{aligned}$$

Each efficient solution $\hat{x}$ of PHMOP$_p(t)$ defines a set of infinitely many efficient solutions $X = \{\lambda\hat{x} : \lambda > 0\} \subseteq X_E$ of PHMOP. We consider the special case of $p = 2$ next.

Corollary 1. *For $p = 2$ there exists $\hat{y}^1, \hat{y}^2 \in \mathbb{R}^2$ with $\hat{y}^1_1 > 0$ and $\hat{y}^2_1 < 0$ such that $Y_N \subseteq \{0\} \cup \{\lambda\hat{y}^1 : \lambda > 0\} \cup \{\lambda\hat{y}^2 : \lambda > 0\}$.*

Proof. If Y_N is empty or $Y_N = \{0\}$, then the result is true (for any choice of $\hat{y}^1, \hat{y}^2$). If there is a non-dominated point $\hat{y} \neq 0$ with $\hat{y}_1 > 0$ then let $\hat{y}^1 = \hat{y}$; otherwise let $\hat{y}^1 = (1, -1)$. If there is a non-dominated point $\hat{y}$ with $\hat{y}_1 < 0$ then let $\hat{y}^2 = \hat{y}$; otherwise let $\hat{y}^2 = (-1, 1)$. Now choose any element $y \in Y_N$ with $y \neq 0$. If $y_1 > 0$ then y must lie in $\{\lambda\hat{y}^1 : \lambda > 0\}$. Otherwise it would either dominate a point in $\{\lambda\hat{y}^1 : \lambda > 0\}$ contradicting Proposition 1, or be dominated by a point in $\{\lambda\hat{y}^1 : \lambda > 0\}$, contradicting $y \in Y_N$. Similarly if $y_1 < 0$ then y must lie in $\{\lambda\hat{y}^2 : \lambda > 0\}$; otherwise it would either dominate a point in $\{\lambda\hat{y}^2 : \lambda > 0\}$ or be dominated by a point in $\{\lambda\hat{y}^2 : \lambda > 0\}$. This proves the result. □

According to Corollary 1, the entire set of non-dominated points of multi-objective optimisation problem PHMOP with $p = 2$ is determined by optimal solutions of two single-objective optimisation problems PHMOP$_p(t)$ with $t = t_1 < 0$ and $t = t_2 > 0$. To find efficient solutions for the constrained multi-objective optimisation problem PHMOP$_p(t)$ in the case $p > 2$, we propose a procedure based on the well-known ε-constraint method [7]. This method attempts to find an efficient solution by minimising one of the original objectives, while the remaining objectives are transformed to constraints. This gives rise to single-objective constrained optimisation problems of the form:

$$\begin{aligned} &\text{PHMOP}_{p,j}(t, \varepsilon): \\ &\quad \underset{x\in\mathbb{R}^n}{\text{minimise}}\ f_j(x) \\ &\quad \text{subject to } f_p(x) = t \\ &\qquad\qquad f_k(x) \leqq \varepsilon_k \quad \text{for } k = 1, \ldots, p-1; \\ &\qquad\qquad\qquad\qquad\qquad k \neq j, \end{aligned}$$

where $j \in \{1, 2, \ldots, p-1\}$, and $\varepsilon \in \mathbb{R}^{p-1}$ is a vector of $p-1$ bounds. Note that the j'th element of ε, ε_j, is not used by PHMOP$_{p,j}(t, \varepsilon)$, but is assumed to exist as this simplifies the development of Algorithm 1 presented below.

Proposition 2. *Let f be a vector of positively homogeneous functions f_k, $k = 1, \ldots, p$. If $\hat{x}$ is an optimal solution of problem PHMOP$_{p,j}(t, \varepsilon)$ then $\lambda\hat{x}$ with $\lambda > 0$ is an optimal solution of problem PHMOP$_{p,j}(\lambda t, \lambda\varepsilon)$.*

Proof. Since f is a vector of positively homogeneous functions we have $f_k(\lambda\hat{x}) = \lambda f_k(\hat{x}) \leqq \lambda\varepsilon_k$ for all $k = 1, \ldots, p-1, k \neq j$ and $f_p(\lambda\hat{x}) = \lambda f_p(\hat{x}) = \lambda t$, so that $\lambda\hat{x}$ is feasible for problem PHMOP$_{p,j}(\lambda t, \lambda\varepsilon)$.

Now, suppose there exists $x \in \mathbb{R}^n$ that is feasible for PHMOP$_{p,j}$ $(\lambda t, \lambda\varepsilon)$ such that

$$f_j(x) < f_j(\lambda\hat{x})$$

for some j. Then x/λ is a feasible solution for problem PHMOP$_{p,j}(t, \varepsilon)$ and

$$f_j\left(\frac{x}{\lambda}\right) < \frac{1}{\lambda}f_j(\lambda\hat{x}) = f_j(\hat{x}).$$

Hence $\hat{x}$ is not an optimal solution of PHMOP$_{p,j}(t, \varepsilon)$, a contradiction. □

It is well known that any optimal solution $\hat{x}$ of PHMOP$_{p,j}(t, \varepsilon)$ is at least a weakly efficient solution of PHMOP$_p(t)$ (there is no x such that $f'(x) < f'(\hat{x})$) and that if $\hat{x}$ is a unique optimal solution of PHMOP$_{p,j}(t, \varepsilon)$ then it is an efficient solution of PHMOP$_p(t)$. Moreover, each efficient solution of PHMOP$_p(t)$ is also an optimal solution of PHMOP$_{p,j}(t, \varepsilon)$ with appropriately chosen j and ε. For more details on the ε-constraint method see, e.g. [3].

To identify efficient solutions of PHMOP$_p(t)$ we solve PHMOP$_{p,j}(t, \varepsilon)$ for different j with a variety of vectors of bounds $\varepsilon \in \mathbb{R}^{p-1}$. To this end we first estimate the range $[l_k, u_k]$ of values that objective f_k takes over efficient solutions of PHMOP$_p(t)$. The exact ranges of values are defined by the ideal point $y^I_k := \min\{f_k(x) : x \in \mathbb{R}^n, f_p(x) = t\}$ and nadir point $y^N_k := \max\{f_k(x) : x$ is an efficient solution of PHMOP$_p(t)\}$. So we set $l_k = y^I_k$. Unfortunately, y^N is very difficult to compute and usually not available, see e.g. [5], and so we suggest the following heuristic approach based on lexicographic optimisation. Let Π denote the set of all permutations of $\{1, \ldots, p-1\}$. For a specific permutation $\pi \in \Pi$ we denote its k'th element by $\pi(k)$. Lexicographic optimisation chooses a permutation π of objectives and optimises them sequentially. First $f_{\pi(1)}(x)$ is optimised over $x \in \mathbb{R}^n$ subject to $f_p(x) = t$. Then, $f_{\pi(2)}(x)$ is optimised over $x \in \mathbb{R}^n$ subject to $f_p(x) = t$ and an equality constraint on the optimal value of $f_{\pi(1)}(x)$. This process continues until an optimal solution of $f_{\pi(p-1)}(x)$ is obtained, giving a final solution we denote $\hat{x}^\pi$. It is clear that $\hat{x}^\pi$ is an efficient solution of PHMOP$_p(t)$. We then set

$$u_k := \max\left\{f_k(\hat{x}^\pi) : \pi \in \Pi\right\}. \tag{1}$$

Clearly, $u_k \leqq y^N_k$. Moreover, for $p = 2$, $u_k = y^N_k$ (see e.g. [5]).

Having found $[l_j, u_j]$ for each objective $j \in \{1, \ldots, p-1\}$, we create a set of (typically equally spaced) points $X_j = \{x^1_j, x^2_j, \ldots, x^{|X_j|}_j\}$ over $[l_j, u_j]$ for each j. For each objective $j = 1, 2, \ldots, p-1$, we then form a grid of points over the other objectives, giving $\mathcal{E}_j = X_1 \times X_2 \times \cdots \times X_{j-1} \times \{\infty\} \times X_{j+1} \times \cdots \times X_{p-1}$, where each $\varepsilon \in \mathcal{E}$ can serve as a vector of bounds in PHMOP$_{p,j}(t, \varepsilon)$. (Note that ε_j is initially unused, but the $\varepsilon_j = \infty$ value will be updated subsequently.) We then solve PHMOP$_{p,j}(t, \varepsilon)$ for each $\varepsilon \in \mathcal{E}_j$. As mentioned before, an optimal solution of this problem could be weakly efficient rather than efficient. To ensure that an

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