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Stochastic comparisons of component and system redundancies with dependent components

Nitin Gupta*, Somesh Kumar

Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India

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ABSTRACT

Under the assumption of dependent identically distributed components and redundant (spares) components, the problem of stochastic comparison of component and system redundancies have been considered. This study has been carried out under the criteria of the likelihood ratio ordering, the reversed failure rate ordering, the failure rate ordering and the usual stochastic ordering.

1. Introduction

In the field of reliability engineering and survival analysis studying the reliability characteristics of coherent systems play an important role. To enhance the reliability characteristics of a coherent system, redundant (spares) components may be provided to its components. These components may be provided to components of the coherent system using active redundancy. Active redundancy means spares have been attached in parallel to the components of the coherent system, i.e., the component and spare take the maximum of the random lifetimes. This active redundancy may be used to provide spares to the coherent system either as component redundancy or system redundancy. In component redundancy for each component, an active spare is provided in the coherent system; while in the system redundancy, coherent system is duplicated and attached as an active redundant spare to the original coherent system. It is well known in reliability engineering that the lifetime of a coherent system having active spare allocation at the component level dominates a coherent system having active spare allocation at the system level, under the usual stochastic ordering. The problem of stochastic comparison of component and system redundancies have been considered by many researchers; see for example, Barlow and Proschan [1], Boland and El-Neweihi [2], Singh and Singh [14], Gupta and Nanda [6], Misra, Dhariyal and Gupta [8], Brito, Zequeira and Valdés [3] and references cited therein.

Consider a random variable X having the probability density function $f(x),\ x \in \mathbb{R} = (-\infty, \infty)$, the distribution function $F(x) = P(X \le x),\ x \in \mathbb{R}$ and the survival function $\bar{F}(x) = 1 - F(x) = P(X > x),\ x \in \mathbb{R}$. The terms increasing/decreasing will be used for monotone non-decreasing/non-increasing throughout the presentation. The symbol $\wedge$ and $\vee$ denote the minimum and maximum, respectively. For the completeness in the presentation, some definitions are given below (see for example Shaked and Shanthikumar [13]):

Definition 1. Let Z_1 and Z_2 be two random variables with Z_i having probability density function $h_i(\cdot)$, distribution function $H_i(\cdot)$, survival function $\bar{H}_i(\cdot) = 1 - H_i(\cdot)$, failure rate function $r_{Z_i}(x) = h_i(x)/\bar{H}_i(x)$ and reversed failure rate function $\tilde{r}_{Z_i}(x) = h_i(x)/H_i(x)$, $i = 1, 2$. Suppose that Z_1 and Z_2 have the common support $[0, \infty)$, and $h_i(x) \ge 0, \forall x \ge 0,\ i = 1, 2$. Then Z_1 is said to be smaller than Z_2 in the

(a) likelihood ratio (lr) order ($Z_1 \le_{\rm lr} Z_2$) if $h_2(x)/h_1(x)$ increases in x;
(b) failure rate (fr) order ($Z_1 \le_{\rm fr} Z_2$) if $r_{Z_1}(x) \ge r_{Z_2}(x)$, for all x;
(c) reversed failure rate (rfr) order ($Z_1 \le_{\rm rfr} Z_2$) if $\tilde{r}_{Z_1}(x) \le \tilde{r}_{Z_2}(x)$, for all x;
(d) usual stochastic (st) order ($Z_1 \le_{st} Z_2$) if $\bar{H}_1(x) \le \bar{H}_2(x)$, for all x.

* Corresponding author.
E-mail addresses: nitin.gupta@maths.iitkgp.ernet.in, nitinstat@gmail.com (N. Gupta), smsh@maths.iitkgp.ernet.in (S. Kumar).

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Following equivalences are easy to verify:

$$Z_1 \leq_{\rm lr} Z_2 \quad \Leftrightarrow \frac{h_2(x)}{h_1(x)} \text{ is increasing in } x \in (0, \infty),$$
$$\Leftrightarrow \bar{H}_2\left(\bar{H}_1^{-1}(p)\right) \text{ is concave in } p \text{ on } (0, 1),$$
$$\Leftrightarrow \bar{H}_1\left(\bar{H}_2^{-1}(p)\right) \text{ is convex in } p \text{ on } (0, 1),$$
$$\Leftrightarrow H_2\left(H_1^{-1}(p)\right) \text{ is convex in } p \text{ on } (0, 1),$$
$$\Leftrightarrow H_1\left(H_2^{-1}(p)\right) \text{ is concave in } p \text{ on } (0, 1);$$
$$Z_1 \leq_{\rm fr} Z_2 \quad \Leftrightarrow \frac{\bar{H}_2\left(\bar{H}_1^{-1}(p)\right)}{p} \text{ is decreasing in } p \text{ on } (0, 1),$$
$$\Leftrightarrow \frac{\bar{H}_1\left(\bar{H}_2^{-1}(p)\right)}{p} \text{ is increasing in } p \text{ on } (0, 1);$$
$$Z_1 \leq_{\rm rfr} Z_2 \quad \Leftrightarrow \frac{H_2\left(H_1^{-1}(p)\right)}{p} \text{ is increasing in } p \text{ on } (0, 1),$$
$$\Leftrightarrow \frac{H_1\left(H_2^{-1}(p)\right)}{p} \text{ is decreasing in } p \text{ on } (0, 1).$$

Following implications are also available in the literature:

$Z_1 \leq_{\rm lr} Z_2 \Rightarrow Z_1 \leq_{\rm fr} Z_2 \Rightarrow Z_1 \leq_{st} Z_2$;

$Z_1 \leq_{\rm lr} Z_2 \Rightarrow Z_1 \leq_{\rm rfr} Z_2 \Rightarrow Z_1 \leq_{st} Z_2$.

For these definitions and other equivalences/implications, reader is referred to Shaked and Shanthikumar [13].

Here we provide a brief review of the results available in the literature. These results are related to the results proved in this paper.

For independently and identically distributed (IID) components and spares, Boland and EI-Neweihi [2], conjectured that for r-out-of-n systems, the component level active redundancy is better than the system level active redundancy with respect to the failure rate ordering. Singh and Singh [14] resolved this conjecture by proving a stronger result that for r-out-of-n systems, the component level active redundancy is better than the system level active redundancy with respect to the likelihood ratio ordering. Misra, Dhariyal and Gupta [8] provided the necessary and sufficient conditions on the structure function under which the component level active redundancy is better than the system level active redundancy with respect to the likelihood ratio ordering. As a corollary Misra, Dhariyal and Gupta [8] obtain the result of Singh and Singh [14].

In real life situations the components which are joined together to form a system may not be independent. The dependency between components comes from environmental factors, the location of the component in the system etc. For example Ghoraf [5] has considered a system to be Markovian, i.e., failure probability of a given component depends upon the state of the preceding component in the system. He also provided various practical examples where dependency comes into the picture. Yang [19] has also studied the reliability analysis of repairable systems with dependent components.

Consider a scalable inertial reference unit for space which is composed of different components which may be dependent. In a space mission it is not possible to repair the unit. Hence it may be desirable to increase the reliability of the unit by component/system redundancy. The spares which are attached to the system become dependent due to the design and placement of the spare in the system. Therefore it is useful to study the comparison of component and system redundancies having dependent identically distributed components and spares.

Considering that components and spares be dependent and identically distributed (DID), in Section 2, we provide necessary and sufficient conditions on the system reliability/survival functions so that the lifetime of a component level active redundancy is better/worse than the lifetime of a system level active redundancy with respect to various stochastic orderings. Some examples are provided to illustrate the results.

2. Comparison of component and system redundancies

Consider a coherent system, with structure function ϕ, having n dependent components $C_1, \ldots, C_n$. Let $X_1, \ldots, X_n$ denote the random lifetimes of these n dependent components $C_1, \ldots, C_n$, respectively. Further $\tau(\mathbf{X}) = \tau(X_1, \ldots, X_n)$ denote the lifetime of the coherent system ϕ. Consider that these n dependent components $C_1, \ldots, C_n$ have the common probability density function $f(\cdot)$, the common distribution function $F(\cdot)$ and the common survival function $\bar{F}(\cdot) = 1 - F(\cdot)$, where $F(0) = 0$. The joint reliability/survival function of $(X_1, \ldots, X_n)$ is given by

$$\bar{G}(x_1, \ldots, x_n) = P(X_1 > x_1, \ldots, X_n > x_n) = K(\bar{F}(x_1), \ldots, \bar{F}(x_n)), \tag{2.1}$$

here $K(0, \ldots, 0) = 1$; the representation (2.1) is called Sklar's copula representation. The function K is known as reliability/survival copula. It is known that the multivariate distribution function K has uniform marginal distributions on $(0, 1)$.

Sklar's copula representation separates out the dependence structure from the marginal distributions for studying multivariate distributions. Conversely, we can construct multivariate distributions using marginal distributions and a copula. Therefore Sklar's copula representation helps us to study different copulas (dependence structures) while retaining marginal distributions. There are various copulas available in the literature such as Archimedean copula, Independence copula, Clayton-Oakes (CO) copula etc. We refer the reader to Nelson [11] and Kulgman et al. [7] for a detailed discussion on the copulas.

The system reliability/survival function of lifetime $\tau(\mathbf{X}) = \tau(X_1, \ldots, X_n)$ is given by

$$\bar{G}_T(x) = P(\tau(\mathbf{X}) > x), \quad x \in \mathbb{R}.$$

The following theorem by Navarro et al. [9] (see also Navarro et al. [10]) provides an important representation of system reliability/survival function $\bar{G}_T(x)$ as a distorted function of the common reliability function $\bar{F}(x)$:

Lemma 1. *Let $\tau(\mathbf{X})$ be the lifetime of a coherent system with DID component lifetimes $X_1, \ldots, X_n$ having common reliability/survival function $\bar{F}(\cdot)$. Then the system reliability/survival function can be written as*

$$\bar{G}_T(x) = h\left(\bar{F}(x)\right),$$

where h depends only on structure function ϕ and the survival copula K of $X_1, \ldots, X_n$. Moreover h is an increasing function on $(0, 1)$, such that $h(0) = 0$ and $h(1) = 1$.

We refer the reader to Denneberg [4], Quiggin [12], Sordo et al. [15], Wang [16,17], Wang and Young [18] and Yarri [20] for a detailed study of distorted distributions/functions and their applications.

Consider the n dependent spares $R_1, \ldots, R_n$ having lifetimes $Y_1, \ldots, Y_n$, with the common probability density function $f(\cdot)$, the common distribution function $F(\cdot)$ and the common survival function $\bar{F}(\cdot) = 1 - F(\cdot)$, where $F(0) = 0$. The components $C_1, \ldots, C_n$ and spares $R_1, \ldots, R_n$ may be dependent, but clearly under the assumption of being identically distributed.

In component redundancy we allocate to each component C_i of coherent system an active spare R_i, $i = 1, \ldots, n$. Then the resulting coherent system, denoted by S_C having DID component lifetimes $X_1, \ldots, X_n$ and spares lifetimes $Y_1, \ldots, Y_n$, has lifetime denoted by

$$\tau(\mathbf{X} \vee \mathbf{Y}) = \tau(X_1 \vee Y_1, \ldots, X_n \vee Y_n).$$

The system S_C has the survival function

$$\bar{F}_C(x) = h\left(1 - (1 - \bar{F}(x))^2\right), \quad x \in \mathbb{R} \tag{2.2}$$

and the probability density function

$$f_C(x) = 2f(x)\left(1 - \bar{F}(x)\right) h'\left(1 - (1 - \bar{F}(x))^2\right), \quad x \in \mathbb{R}.$$