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Dynamic portfolio selection with market impact costs

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ABSTRACT

This paper concerns optimal dynamic portfolio choice with quadratic utility when there are market impact costs. The optimal policy is difficult to characterize, so we look instead for sub-optimal policies. Our proposed suboptimal policy solves a tractable dynamic portfolio choice problem where the cost of trading is captured in the objective instead of the price dynamics. A multiple time scale asymptotic expansion shows that our proposed policy has sensible structural properties, while numerical experiments show promising performance and robustness properties.

1. Introduction

Periods of illiquidity can make it difficult for an investor to trade a large quantity of an asset within a desired period of time. Much of the recent effort in modeling illiquidity and accounting for its effects in hedging, portfolio choice, and trade execution comes from the recognition that ignoring it is both risky and costly (e.g. Bertsimas and Lo [4], Almgren and Chriss [2], Grinold [6] and Garleanu and Pedersen [5], He and Mamaysky [7], Ly Vath et al. [14], Moallemi and Saglam [11], Rogers and Singh [12] and Schied et al. [13]). While much of this literature accounts for illiquidity by explicitly modeling the impact of trades on asset prices (exceptions include Grinold [6], Garleanu and Pedersen [5] and Moallemi and Saglam [11], on dynamic active portfolio management), this approach is challenging because even simple models of price impact lead to substantially harder optimization problems, and it is difficult to see how the techniques which have been used extend to multi-assets, or to problems that include features such as stochastic liquidity, etc.

We are interested in optimal utility maximization when there are market impact costs. In this paper, we take a different approach where instead of searching for *the* optimal policy, which is difficult when there is market impact, we look for a sub-optimal policy that is "good enough". The novelty is that our suboptimal policies solve an alternative tractable dynamic portfolio choice problem which captures an essential tradeoff between *trading slowly* (to minimize market impact) while having *desirable asset holdings across time* (to achieve good returns). Our model achieves tractability by modeling market impact through a penalty term in the objective that penalizes rapid trading. This differs from the standard approach where market impact is explicitly modeled in the price dynamics. We show using multiple time scale asymptotic methods that the portfolios obtained by solving our model have sensible structural properties. Simulations suggest that it delivers close-to-optimal utility when applied to the price impact model of Almgren et al. [3].

Outline:

We present the Merton model of dynamic portfolio choice in perfectly liquid markets in Section 2, and an extension with price impact, based on ideas from Almgren et al. [3], in Section 3. While it is not possible to solve this problem analytically, its construction reveals an essential tradeoff is between maintaining desirable portfolio holdings over time to optimize returns, and minimizing the cost of doing so. With this in mind, we formulate a surrogate dynamic portfolio choice problem in Section 4 where illiquidity costs are accounted for in the objective instead of the price dynamics. The benefit of this model is that it captures this tradeoff while retaining tractability. We establish the relationship between our surrogate model, the perfectly liquid Merton problem, and another portfolio selection problem with trading costs, in Section 5. This relationship and multiple time scale asymptotic expansions are used in Section 6 to derive an expansion of the

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optimal trading rate in the regime of vanishing liquidity costs, which is used to understand comparative statics and structural properties of the policy obtained from our surrogate model. In Section 7, we evaluate the performance and robustness properties of our portfolio on the temporary price impact model that we formulated in Section 3.

2. Portfolio selection problem in liquid market

We recall the classical Merton problem [10] for frictionless markets.

Asset dynamics

For simplicity, we consider a market with one risky asset and one risk-free asset. Our results can be extended to multiple assets with no essential difficulty. We model uncertainty using Brownian motion which is assumed to live on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ over a finite time horizon $[0, T]$. The risky asset price $s(t)$ is assumed to follow geometric Brownian motion

$$ds(t) = \mu s(t)dt + \sigma s(t)dw(t), \tag{1}$$

with expected return μ and volatility σ. The risk-free asset price process $s_0(t)$ satisfies

$$ds_0(t) = rs_0(t)dt \tag{2}$$

with risk-free rate of return r.

The Merton problem

Let $x(t)$ denote the investor's wealth and $\pi(t)$ be the value of his/her risky asset holding at time t. The classical Merton problem maximizes expected utility of terminal wealth

$$\begin{cases} \sup\limits_{\pi(\cdot)} \ E\{\Phi(x(T))\} \\ \text{subject to:} \\ dx(t) = \{x(t)r + \pi(t)(\mu - r)\}dt + \pi(t)\sigma dw(t) \\ x(0) = x_0. \end{cases} \tag{3}$$

Explicit solutions for the Merton problem can be found when the utility function is of power, exponential, logarithmic and quadratic type. In the case of quadratic utility we have the following result, which can be shown by solving the associated dynamic programming equations.

Proposition 1. *The value function for the Merton problem (3) with quadratic utility function $\Phi(x(T)) = x(T) - \frac{\eta}{2}x(T)^2$ is*

$$V_M(t, x) = -\frac{1}{2}A_M(t)x^2 + B_M(t)x + C_M(t) \tag{4}$$

where

$$A_M(t) = \eta e^{\left(2r - \frac{(\mu-r)^2}{\sigma^2}\right)(T-t)},$$

$$B_M(t) = e^{\left(r - \frac{(\mu-r)^2}{\sigma^2}\right)(T-t)},$$

$$C_M(t) = \frac{1}{2\eta}(1 - e^{-(\frac{\mu-r}{\sigma})^2(T-t)}).$$

The optimal investment policy is

$$\pi_M^*(t, x) = \frac{\mu - r}{\sigma^2}\left(\frac{B_M(t)}{A_M(t)} - x\right). \tag{5}$$

(Note that the subscript M is added for later reference.)

3. Dynamic portfolio choice with market impact costs

Consider an investor who rebalances daily over a finite time horizon T. We denote the rebalancing time and rebalancing interval by n and Δ (in years), respectively. The market consists of a risk-free asset and one risky asset. The risk-free asset follows the dynamics (2). We adopt a temporary impact model for the risky asset that builds on Almgren et al. [3], which we now describe.

The risky asset price is described using two components, an *observed price* $s(n)$ and an *execution price* $s^{exec}(n, s(n), N(n))$. The execution price $s^{exec}(n, s(n), N(n))$ is the average price of each asset for an order of size $N(n)$ submitted at the start of period n after observing a price of $s(n)$. The execution price $s^{exec}(n, s(n), N(n))$ is larger than the observed price $s(n)$ when buying an asset and smaller when selling, and can be viewed as the cost of moving through an order book in order to execute a block trade. We assume in this paper that observed price $s(n)$ is geometric Brownian motion (1) sampled at daily intervals, and that the execution price is given by

$$s^{exec}(n, s(n), N(n)) = s(n)(1 + J(N(n))). \tag{6}$$

Here, $J(N(n))$ is the *relative price impact* function which is positive when the investor buys the asset ($J(N) \geq 0$ when $N > 0$) and negative when selling ($-1 < J(N) \leq 0$ when $N < 0$). Almgren et al. [3] use the function

$$J(N(n)) = c \ \text{sgn}(N(n)) \left(\frac{|N(n)|}{V}\right)^{0.6} \tag{7}$$

where c and V are constants representing market depth and average daily volume, respectively. Observe that the sign of $J(N(n))$ depends on trading direction and its magnitude is proportional to the size of the trade to the power of 0.6 (the value fitted in [3] using data).

We denote by $x(n) \triangleq \pi(n) + y(n)$ the investor's wealth at the start of time period n. Here $\pi(n)$ denotes the dollar value of the investor's investment in the risky asset, which we *define* as $\pi(n) \triangleq \psi(n)s(n)$ where $\psi(n)$ is the number of shares that he owns of the risky asset, and $y(n)$ is the dollar value of his investment in the risk-free asset.

At the start of period n, the investor observes the risky asset price $s(n)$, the value of his risky asset holding $\pi(n)$, and his wealth $x(n)$. On the basis of this information, he trades $N(n)$ shares at price $s^{exec}(n, s(n), N(n))$. The value of his risk-free holding immediately after the trade equals its pre-trade value net the cost of this transaction

$$\begin{aligned} y(n^+) &= y(n) - N(n)s^{exec}(n, s(n), N(n)) \\ &= x(n) - \pi(n) - N(n)s^{exec}(n, s(n), N(n)), \end{aligned} \tag{8}$$

while the value of the risky holding increases from its pre-trade amount by the value of the assets just added

$$\pi(n^+) = \pi(n) + N(n)s(n). \tag{9}$$

A key element of the model (8)–(9) is that it explicitly models the cost of trading and the impact of these costs on the price dynamics and the investor's wealth. For instance, if $N(n)$ assets are purchased at the start of period n, a cash amount of $N(n)s^{exec}(n, s(n), N(n))$ is taken from his risk-free holdings to pay for these assets (perhaps by moving through the order book). These assets are then added to his risky asset holdings, adding value $N(n)s(n)$. The cost of acquiring these assets is the difference $N(n)[s^{exec}(n) - s(n)]$. (A similar argument shows that $N(n)[s(n) - s^{exec}(n)]$ is the cost to the investor when $N(n)$ assets are sold.) In both cases, the cost of trading equals $|N(n)|s(n)|J(N(n))|$. For the relative price impact