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On some conditional characteristics of hazard rate processes induced by external shocks

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ABSTRACT

Stochastic failure models for systems under randomly variable environment (dynamic environment) are often described using hazard rate process. In this paper, we consider hazard rate processes induced by external shocks affecting a system that follow the nonhomogeneous Poisson process. The sample paths of these processes monotonically increase. However, the failure rate of a system can have completely different shapes and follow, e.g., the upside-down bathtub pattern. We describe and study various 'conditional properties' of the models that help to analyze and interpret the shape of the failure rate and other relevant characteristics.

1. Introduction

Many of the currently used failure models have been developed on the premise that the operating environment is static. However, devices often work in varying environments and, therefore, their performance can be significantly affected by varying environmental conditions. Stochastic failure models that include a time-varying environment can generally be classified into two broad categories. The first category usually employs 'hazard rate processes' (Aven and Jensen [4,5]) as a method of describing environmental stochasticity. Some initial explicit results can be found in Gaver [16], Arjas [3], Kebir [19], Gamerman [15] and Lemoine and Wenocur [26]. Later on, Banjevic et al. [7] assumed that the environment covariate is driven by a Markov process and used an approximation scheme to estimate the corresponding failure time distribution. Some computational issues for calculating the reliability function for such systems were addressed by Banjevic and Jardine [6]. More recently, flexible proportional hazard type model has been widely used to relate the hazard rate function of a component's lifetime to environmental conditions (see, e.g., Banjevic et al. [7], Jardine et al. [17], Lee and Whitmore [24] and Liao and Tseng [28]). Zhao et al. [31] in this way discussed condition-based inspection policies for systems subject to random shocks.

On the other hand, the second category includes a class of stochastic models that describe degradation of systems directly using different stochastic processes such as Brownian motion or general diffusions, Lévy processes, Markov renewal models and random coefficient models. Doksum and Høyland [12] used Brownian motion with a stress-dependent drift parameter to derive the failure time distribution. Kharoufeh [20] and Kharoufeh and Cox [21] examined a model for a system degrading linearly at a rate that depends on the state of a continuous-time Markov chain. In Kharoufeh and Mixon [23], a model with Markov-modulated degradation rates and Poisson shock intensities was studied. See also Liao and Tseng [28], Kharoufeh et al. [22], Anderson [2], Li and Anderson [27] for related models.

In this paper, we follow the 'hazard rate process approach', which, in our opinion, has not been sufficiently investigated. Thus, we will consider the failure rate as a specific increasing stochastic process $\{r_t,\ t \geq 0\}$ and, in this way, describe stochastic aging in an 'aggregated form'.

Now we are ready to describe the specific setting of our interest. Assume that a system, whose lifetime is denoted by T, is operating in a random environment described by a certain (covariate) stochastic process $\{Z(t),\ t \geq 0\}$. For example, the stochastic process $\{Z(t),\ t \geq 0\}$ can represent the randomly changing time-dependent external temperature, electric or mechanical load, or some other randomly changing external stress, etc. Then, the

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conditional failure rate can formally be defined (see Kalbfleisch and Prentice [18], Aalen et al. [1]) as

$$r(t \mid z(u), 0 \le u \le t) \equiv \lim_{\Delta t \to 0} \frac{P(t < T \le t + \Delta t \mid Z(u) = z(u),\ 0 \le u \le t,\ T > t)}{\Delta t}.$$

Note that this conditional failure rate is specified for a realization of the covariate process. With the covariate process not fixed yet, as discussed above, it is obviously the hazard rate process. Under certain non-restrictive and technical assumptions on this process, the following exponential representation for its realizations exists (see Lehmann [25] for details):

$$P(T > t \mid Z(u) = z(u), 0 \le u \le t) = \exp\left\{-\int_0^t r(s \mid z(u),\ 0 \le u \le s) ds\right\}. \quad (1)$$

We will now describe the lifetime distribution of our system. In our model, the random external environment is modeled by the external shock process $\{N(t),\ t \ge 0\}$, where $N(t)$ represents the total number of shocks by time t, and the related marked process. We will assume that the external shock process $\{N(t),\ t \ge 0\}$ is the nonhomogeneous Poisson process with intensity function $\lambda(t)$. Also, denote by $T_1 \le T_2 \le \cdots$ the sequential arrival times of external shocks. Let $\Psi_1, \Psi_2, \ldots$ be i.i.d. random sequence of continuous random variables, having common Cdf $G(t)$. Assume that the conditional failure rate function of T for our system is Eq. (2) (given in Box I), where $r_0(t)$ is the 'baseline failure rate' which defines the lifetime distribution under the laboratory environment, i.e., when there is no external shock process. From (2) the effect of external shocks on the lifetime of T can be read as follows: 'on ith shock, the failure rate of T is increased by ψ_i'. (See also Nakagawa [29], Cha and Lee [10], Cha and Mi [11], Cha and Finkelstein [8,9] and Finkelstein and Cha [14] for various shock models.)

The description of stochastic failure model based on the conditional failure rate (2) will allow some meaningful interpretations of the behavior of the 'unconditional' failure rate of the system, which can be of a major interest. As mentioned before, with the external shock process not fixed yet, the conditional failure rate (2) is a stochastic process, $\{r_t,\ t \ge 0\}$, of the form:

$$r_t \equiv r(t \mid N(u),\ 0 \le u \le t; \Psi_i,\ i = 1, 2, \ldots, N(t)) = r_0(t) + \sum_{i=1}^{N(t)} \Psi_i. \quad (3)$$

Each realization of r_t in (3) is an 'ordinary failure rate', obviously conditioned on survival event $T > t$. However, random quantities in (3), i.e., $N(u)$ and Ψ_i, are not conditioned on survival, and therefore cannot objectively describe 'dynamics' for the corresponding realizations. Therefore, in principle, the corresponding conditional process should be better called the 'hazard rate process'. However, in line with the existent terminology, we will retain this term for $\{r_t,\ t \ge 0\}$, whereas $\{(r_t \mid T > t),\ t \ge 0\}$ will be referred to as the 'conditional hazard rate process'.

A similar setting is defined by the following additive fixed frailty model that describes heterogeneous population consisting of homogeneous subpopulations ordered in the sense of the hazard rate ordering:

$$r(t \mid Z = z) = r_0(t) + z,$$

where $Z \ge 0$ is the frailty parameter with the pdf $\pi(z)$. In this case, as the weakest subpopulations are dying out first, the composition of the population (dynamics) changes with time and is described by the corresponding conditional distribution $(Z \mid T > t)$ for each t (Finkelstein [13]), which is the simpler analogue of our $\{(r_t \mid T > t),\ t \ge 0\}$. In our case, there is no ordering of the sample paths of the hazard rate process, however, the most 'vulnerable' realizations are still 'dying out first' thus defining the composition of survivors as the function of time.

In order to describe the corresponding distribution for survivors for model (3) and to obtain the unconditional (ordinary) failure rate for a system operating in a random environment of the described type, we must consider the conditional joint distribution of $(N(t), \Psi_i,\ i = 1, \ldots, N(t) \mid T > t)$.

2. The conditional distribution of $(N(t), \Psi_i,\ i = 1, \ldots, N(t) \mid T > t)$ and the failure rate function

In accordance with model (3), the unconditional failure rate of the system, which is denoted by $r(t)$, can be derived as Eq. (4) (given in Box II), where $E_{N(t),\Psi_i,\ i=1,2,\ldots,N(t)|T>t}$ stands for the expectation with respect to the conditional distribution of $(N(t), \Psi_i,\ i = 1, 2, \ldots, N(t) \mid T > t)$. As the unconditional failure rate in (4) contains the conditional expectation $E\left[\sum_{i=1}^{N(t)} \Psi_i \mid T > t\right]$, it is necessary to derive the conditional distribution of $(N(t), \Psi_i,\ i = 1, 2, \ldots, N(t) \mid T > t)$ and to investigate its behavior in order to interpret the shape of the unconditional failure rate function $r(t)$.

Theorem 1. *Let $M_\Psi(t)$ be the mgf of Ψ_i. The conditional joint distribution of $(\Psi_1, \Psi_2, \ldots, \Psi_{N(t)},\ N(t) \mid T > t)$ is given by*

$$\begin{aligned} &f_{\Psi_1,\Psi_2,\ldots,\Psi_{N(t)},N(t)|T>t}(x_1, x_2, \ldots, x_n, n) \\ &= \left(\prod_{i=1}^{n} \frac{\int_0^t \exp\{-x_i(t-v)\} g(x_i)\lambda(v)dv}{\int_0^t \int_0^\infty \exp\{-x(t-v)\} g(x)dx\lambda(v)dv}\right) \\ &\times \frac{\left(\int_0^t M_\Psi(-(t-v))\lambda(v)dv\right)^n}{n!} \\ &\times \exp\left\{-\int_0^t M_\Psi(-(t-v))\lambda(v)dv\right\}, \\ &x_i \ge 0,\ i = 1, 2, \ldots, n,\ n = 0, 1, 2, \ldots. \end{aligned}$$

Proof. Note that the history of the shock process $\{N(u),\ 0 \le u \le t\}$ can completely be specified by $\{T_1, T_2, \ldots, T_{N(t)}, N(t)\}$. Then, according to the relationship between the conditional failure rate and the conditional survival function stated in Eq. (1), in our model,

$$\begin{aligned} &P(T > t \mid T_1, T_2, \ldots, T_{N(t)}, N(t); \Psi_i,\ i = 1, 2, \ldots, N(t)) \\ &= \exp\left\{-\int_0^t r_0(u)du\right\} \exp\left\{-\int_0^t \sum_{i=1}^{N(u)} \Psi_i du\right\} \\ &= \exp\left\{-\int_0^t r_0(u)du\right\} \exp\left\{-\sum_{i=1}^{N(t)} \Psi_i(t - T_i)\right\} \\ &= \exp\left\{-\int_0^t r_0(u)du\right\} \prod_{i=1}^{N(t)} \exp\{-\Psi_i(t - T_i)\}. \quad (5) \end{aligned}$$

For a more convenient mathematical handling of our model, the conditional survival function in (5) can be equivalently stated in terms of 'randomized set of random variables', which will allow us to conveniently handle independent random variables:

$$\begin{aligned} &P(T > t \mid V_1, V_2, \ldots, V_{N(t)}, N(t); \Psi_i,\ i = 1, 2, \ldots, N(t)) \\ &= \exp\left\{-\int_0^t r_0(u)du\right\} \prod_{i=1}^{N(t)} \exp\{-\Psi_i(t - V_i)\}, \quad (6) \end{aligned}$$