Operations Research Letters 42 (2014) 337–342

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Mean–variance portfolio selection under a constant elasticity of variance model

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ARTICLE INFO

Article history:
Received 16 May 2013
Received in revised form
8 May 2014
Accepted 27 May 2014
Available online 4 June 2014

Keywords:
Mean–variance portfolio selection
Constant elasticity of variance model
Backward stochastic Riccati equation
Efficient frontier

ABSTRACT

This paper discusses a mean–variance portfolio selection problem under a constant elasticity of variance model. A backward stochastic Riccati equation is first considered. Then we relate the solution of the associated stochastic control problem to that of the backward stochastic Riccati equation. Finally, explicit expressions of the optimal portfolio strategy, the value function and the efficient frontier of the mean–variance problem are expressed in terms of the solution of the backward stochastic Riccati equation.

1. Introduction

Portfolio selection problem is an important issue in the theory and practice of finance. The modern portfolio selection theory can be traced back to the seminal work of Markowitz [23], where a mean–variance formulation was developed in a single-period setting with the Gaussian assumption for the distributions of individual returns. Ever since then, there has been a growing interest in extending and generalizing Markowitz's work. Using embedding techniques, Li and Ng [20] and Zhou and Li [27] solved the mean–variance portfolio selection problem analytically in a multi-period and continuous-time setting, respectively. Recently, there has been an interest in studying the mean–variance portfolio selection problem in financial models with random parameters. See, for example, Lim and Zhou [22], Ferland and Watier [14] and Chiu and Wong [7], amongst others.

The constant elasticity of variance (CEV) model was first introduced to the financial community by Cox [9]. It may be considered a type of random-coefficient financial models. An empirical advantage of the CEV model is that it can describe the implied volatility smile observed in option prices data. In the last three decades, some works have been done in option valuation under the CEV model. See, for example, Cox and Ross [11], Beckers [2], Davydov and Linetsky [12] and others. Recently, stochastic control problems in insurance and finance under the CEV model have attracted some attention. There were some previous works along this direction such as Xiao et al. [25], Gao [15], Jung and Kim [19], Liang et al. [21], Zhao and Rong [26], and others. However, most of these works focused on the utility maximization problems under the CEV model. It seems that portfolio selection under the CEV model in the Markowitz mean–variance paradigm may have not yet been well-explored.

In this paper, we discuss a continuous-time mean–variance portfolio selection problem with two securities, namely a risk-free bond and a risky share. The price process of the risky share is governed by a CEV model. The financial market described by the CEV model is a complete market with stochastic volatility, which can be regarded as a particular case of the work with general random market parameters in Lim and Zhou [22]. We adopt the stochastic linear-quadratic control approach as in Lim and Zhou [22] and relate the solution of the mean–variance problem under the CEV model to a backward stochastic Riccati equation (BSRE). Although Lim and Zhou [22] established a general theory of the mean–variance portfolio selection problem with random

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http://dx.doi.org/10.1016/j.orl.2014.05.008
0167-6377/

parameters, the solution of the problem depends on solving the BSRE, which is difficult to solve in closed form when an explicit structure of random parameters is not specified. By making use of a particular structure given by the CEV model, a closed-form solution of the BSRE corresponding to the mean–variance portfolio selection problem is derived. Explicit expressions of the optimal portfolio strategy, the value function and the efficient frontier of the mean–variance problem are then represented in terms of the solution of the BSRE.

2. Problem formulation

Let $\mathcal{T}$ be a finite time parameter set $[0, T]$, where $T < \infty$. As usual, a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is considered, where $\mathcal{P}$ is a real-world probability measure and the expectation with respect to $\mathcal{P}$ is denoted as $\mathrm{E}[\cdot]$. Let $\{W(t)|t \in \mathcal{T}\}$ be a one-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. Assume that $\mathbb{F} := \{\mathcal{F}(t)|t \in \mathcal{T}\}$ is the right continuous, $\mathcal{P}$-complete filtration generated by $\{W(t)|t \in \mathcal{T}\}$.

For any nonnegative $\mathbb{F}$-adapted process $\{a(t)|t \in \mathcal{T}\}$, let $\{A(t)|t \in \mathcal{T}\}$ be an increasing continuous process defined by $A(t) := \int_0^t a^2(s)ds$, $t \in \mathcal{T}$. Let $\eta \geq 0$ be a generic constant, which may be different from line to line. On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$, we denote by $\mathcal{L}^2_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re)$ the space of all real-valued, $\mathbb{F}$-adapted processes $\{f(t)|t \in \mathcal{T}\}$ such that $\|f\|^2_\eta := \mathrm{E}[\int_0^T e^{\eta A(t)}|f(t)|^2 dt] < \infty$, by $\mathcal{L}^{2,a}_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re)$ the space of all real-valued, $\mathbb{F}$-adapted processes $\{f(t)|t \in \mathcal{T}\}$ such that $\|f\|^2_{\eta,a} := \|af\|^2_\eta = \mathrm{E}[\int_0^T a^2(t)e^{\eta A(t)}|f(t)|^2 dt] < \infty$, and by $\mathcal{L}^{2,c}_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re)$ the space of all real-valued, $\mathbb{F}$-adapted, càdlàg processes $\{f(t)|t \in \mathcal{T}\}$ such that $\|f\|^2_{\eta,c} := \mathrm{E}[\sup_{0\leq t\leq T} e^{\eta A(t)}|f(t)|^2] < \infty$. Then

$$\mathcal{M}^2_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re \times \Re) := (\mathcal{L}^{2,a}_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re) \cap \mathcal{L}^{2,c}_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re)) \times \mathcal{L}^2_{\mathcal{F}}(\eta, a, \mathcal{T}; \Re),$$

is a Banach space with the norm $\|(y, z)\|^2_\eta := \|y\|^2_{\eta,a} + \|y\|^2_{\eta,c} + \|z\|^2_\eta$. In addition, we denote by $\mathcal{C}^{1,2}(\mathcal{T} \times \Re^+; \Re)$ the space of real-valued continuous functions on $\mathcal{T} \times \Re^+$ with continuous derivative in the first argument and continuous derivatives up to order 2 in the second argument, and by $\mathcal{C}^1(\mathcal{T}; \Re)$ the space of continuously differentiable functions from $\mathcal{T}$ to $\Re$.

We consider a financial market consisting of a risk-free bond and a risky share. The price process of the risk-free bond $\{B(t)|t \in \mathcal{T}\}$ evolves over time as:

$$dB(t) = r(t)B(t)dt, \quad t \in \mathcal{T}, \qquad B(0) = 1, \tag{1}$$

where $r(t)$ represents the risk-free, instantaneous interest rate at time t. Assume that there exists an $\epsilon > 0$ such that $r(t) \geq \epsilon$, for each $t \in \mathcal{T}$.

The price process of the risky share $\{S(t)|t \in \mathcal{T}\}$ satisfies the following stochastic differential equation (SDE):

$$\begin{aligned} &dS(t) = S(t)\big[\mu(t)dt + S^\beta(t)\sigma(t)dW(t)\big], \quad t \in \mathcal{T}, \\ &S(0) = s > 0. \end{aligned} \tag{2}$$

Here, $\mu(t)$ is the appreciation rate of the share; $\sigma(t) > 0$ can be interpreted as the scale parameter of the share; β is called the elasticity parameter of the share. Then, $S^\beta(t)\sigma(t)$ represents the instantaneous volatility of the share at time t. Furthermore, we require that $r(t)$, $\mu(t)$, $\sigma(t)$ are deterministic, uniformly bounded functions of time t.

Throughout this paper, we only consider the case of a negative elasticity parameter (i.e. $\beta < 0$) for two reasons. First, as indicated by Heston et al. [16], there are arbitrage opportunities and asset price bubbles on both option values and share prices in the case that $\beta > 0$. A bubble may be characterized by a price process and, when discounted, is a local martingale under the risk-neutral measure but not a martingale (see, for example, Cox and Hobson [10]). Second, the instantaneous volatility increases as the share price increases in the case of a positive elasticity parameter, which is not consistent with the empirical evidence of the leverage effect (see, for example, Christie [8]).

From Lemma 6.4.4.1 in Jeanblanc et al. [17], the zero boundary is reached almost surely for the CEV model with $\beta < 0$ and it is an absorbing state. Consequently, the CEV model with $\beta < 0$ may be used to describe default. If we assume that the share price process S is killed at the first hitting time of zero and is sent to the absorbing state, the default time is defined as $T_0 := \inf\{t \geq 0|S(t) = 0\}$. Indeed, this kind of default might be related to defaults described by the Merton structural firm value model (see Merton [24]). The structural model is intuitively appealing since it links defaults to the firm's capital structure. However, unlike the reduced-form approach where default times are totally inaccessible, the default time T_0 is predictable with respect to the underlying filtration generated by information about the firm's value. That is, the default event may be predicted by observing the dynamics of the firm's value. This counterfactual feature leads to a discrepancy between the credit spreads from structural models and the market data (see Jones et al. [18]). To overcome the predictability of defaults under the CEV model, one may consider adding a jump-to-default part in the CEV model (2) as in Campi et al. [5] or Carr and Linetsky [6]. Under such a jump-to-default CEV model, defaults could be either expected (predictable) or unexpected (totally inaccessible) depending on whether they are triggered by diffusion and jump terms, respectively. This may provide a possible way to combine the advantages of both structural and reduced-form models for credit risk analysis.

In what follows, we consider the situation where an economic agent invests his wealth in the financial market as described by Eqs. (1) and (2). Let $\pi(t)$ be the amount of the agent's wealth invested in the risky share at time t. Here $\pi(\cdot) := \{\pi(t)|t \in \mathcal{T}\}$ is called a portfolio strategy of the agent. Let $X(t) := X^\pi(t)$ be the total wealth of the agent at time t corresponding to the portfolio strategy $\pi(\cdot)$. Suppose that the portfolio strategy is self-financed. Then the wealth process $\{X(t)|t \in \mathcal{T}\}$ of the agent is governed by the following SDE:

$$\begin{cases} dX(t) = \big[r(t)X(t) + \pi(t)(\mu(t) - r(t))\big]dt \\ \quad + \pi(t)S^\beta(t)\sigma(t)dW(t), \quad t \in \mathcal{T}, \\ X(0) = x. \end{cases} \tag{3}$$

Definition 2.1. A portfolio strategy $\pi(\cdot)$ is said to be admissible if (1) $\pi(\cdot)$ is $\mathbb{F}$-adapted; (2) $\mathrm{E}[\int_0^T \pi^2(t)S^{2\beta}(t)dt] < \infty$; (3) the SDE in Eq. (3) has a unique strong solution $X(\cdot)$ corresponding to $\pi(\cdot)$. The set of all admissible portfolio strategies is denoted by $\mathcal{A}$.

The agent's objective is to find an admissible portfolio $\pi(\cdot) \in \mathcal{A}$ to minimize the variance of terminal wealth for a given level of the expected terminal wealth. Finding such a portfolio $\pi(\cdot)$ is referred to as the mean–variance portfolio selection problem. Specifically, as in the literature, the mean–variance portfolio selection problem is formulated as follows:

Definition 2.2. The mean–variance portfolio selection problem is the following constrained stochastic optimization problem, parameterized by $d \in \Re$:

$$\begin{cases} \min\limits_{\pi(\cdot)\in\mathcal{A}} J(x, s; \pi(\cdot)) = \mathrm{E}[(X(T) - d)^2], \\ \text{subject to } \begin{cases} \mathrm{E}[X(T)] = d, \\ (X(\cdot), \pi(\cdot)) \text{ satisfy (3).} \end{cases} \end{cases} \tag{4}$$