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# An inexact proximal point algorithm for maximal monotone vector fields on Hadamard manifolds[☆]

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ABSTRACT

In this paper, an inexact proximal point algorithm concerned with the singularity of maximal monotone vector fields is introduced and studied on Hadamard manifolds, in which a relative error tolerance with squared summable error factors is considered. It is proved that the sequence generated by the proposed method is convergent to a solution of the problem. Moreover, an application to the optimization problem on Hadamard manifolds is given. The main results presented in this paper generalize and improve some corresponding known results given in the literature.



## 1. Introduction

The theory of maximal monotone operators provides a powerful general framework for the study of convex programming problems and variational inequalities. A basic problem in the theory of maximal monotone operators is to find $x \in \mathbb{R}^n$ such that $0 \in T(x)$, where $T$ is a multivalued maximal monotone operator from $\mathbb{R}^n$ to itself. There is an extensive literature concerning this classical problem (see, for example, [10,23,28,27,26,33]).

In this paper, we will extend an inexact proximal point algorithm with a relative error tolerance from Euclidean spaces to Hadamard manifolds. Thus, we start with an introduction of related inexact proximal point algorithms on Euclidean spaces. The proximal point algorithm is one of the most important methods for solving the original problem, which, starting with any vector $x^0 \in \mathbb{R}^n$, iteratively updates $x^{k+1}$ conforming to the following recursion

$$0 \in c_k T(x^{k+1}) + x^{k+1} - x^k, \tag{1.1}$$

where $\{c_k\} \subset [c, +\infty)$, $c > 0$, is a sequence of scalars. However, as pointed out in [23,10], the ideal form of the method is often impractical, since in many cases, solving problem (1.1) exactly is either impossible or as difficult as solving the original problem $0 \in T(x)$. In [23], Rockafellar gave an inexact variant of the method

$$e^k \in c_k T(\tilde{x}^k) + \tilde{x}^k - x^k, \tag{1.2}$$

where $\{e^k\}$ is regarded as an error sequence. This method is called an inexact proximal point algorithm. Rockafellar [23] provided the following two classes of error criteria:

$$\|e^k\| \le \eta_k \quad \text{with} \sum_{k=0}^{+\infty} \eta_k < +\infty, \tag{1.3}$$

and

$$\|e^k\| \le \eta_k \|\tilde{x}^k - x^k\| \quad \text{with} \sum_{k=0}^{+\infty} \eta_k < +\infty. \tag{1.4}$$

Under suitable assumptions and letting $x^{k+1} = \tilde{x}^k$, Rockafellar [23] proved the global convergence result and locally linear rate of convergence, respectively. Since then, the summability of errors or error factors has been the standard assumption for ensuring the convergence of inexact proximal and proximal-like methods. The error criterions (1.3) and (1.4) can be seen as a absolute error and a relative error, respectively.

From the point of view of numerical analysis, relative errors are easier to estimate and analyze. Therefore, some researchers concentrated their attention on the inexact proximal point algorithms with relative errors. Solodov and Svaiter proposed the following two relative error tolerances:

$$\|e^k\| \le \eta \|\tilde{x}^k - x^k\| \quad \text{with } \eta \in [0, 1) \tag{1.5}$$

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in [26], and

$$\|e^k\| \le \eta \max\{c_k^{-1}\|v^k\|, \|\tilde{x}^k - x^k\|\} \quad \text{with } \eta \in [0, 1) \quad \text{and } v^k \in T(x^k) \tag{1.6}$$

in [27], respectively. Note that the error factor $\eta$ in both relative error tolerances above can be a constant in [0, 1). As a consequence, from the point of view of computation, (1.5) and (1.6) are more attractive than (1.4). However, from two examples supplied by Solodov and Svaiter (see, [26,27]), we know that the traditional inexact proximal point algorithm (i.e., letting $x^{k+1} := \tilde{x}^k$) may not converge under the relative error tolerances (1.5) and (1.6). To ensure the convergence of the proximal point algorithms under (1.5) and (1.6), either extragradient step [26] or projection step [27] was required. Therefore, the two methods proposed by Solodov and Svaiter [26,27] were named as *hybrid extragradient-proximal algorithm* and *hybrid projection-proximal algorithm*, respectively.

Later, without adding an additional *extragradient* or *projection* step to the algorithm, Han and He [10] introduced the following error criterion:

$$\|e^{k+1}\| \le \eta_k \|x^{k+1} - x^k\| \quad \text{with } \sum_{k=0}^{\infty} \eta_k^2 < +\infty. \tag{1.7}$$

It is clear that the error criterion (1.7) is weaker than the one (1.4).

The extension to Riemannian manifolds of the concepts and techniques that fit in Euclidean spaces is natural and nontrivial. Actually, in recent years a large number of researchers have been making great efforts to this topic (see, for example, [1–9,13–15,12,16,18,17,19,21,22,20,32,11,31,29,30]).

In particular, Li et al. [13] studied the proximal point algorithm for singularities (solutions of the inclusion $0 \in A(x)$) of a maximal monotone vector field $A$ on a Hadamard manifold $\mathbb{M}$, which extends the earlier results of Rockafellar [23] from Euclidean spaces to Hadamard manifolds. Its iterative scheme is as follows: given $x^0 \in \mathbb{M}$ and $\{c_k\} \subset [c, +\infty)$, $c > 0$, define $x^{k+1}$ such that

$$0 \in c_k A(x^{k+1}) - \exp^{-1}_{x^{k+1}} x^k. \tag{1.8}$$

Up to now, most of the proximal point algorithms on Riemannian manifolds are exact versions (see, for example, [13,8,4,22,20,30]). However, since the proximal point algorithms are implicit methods in essence, different from projection-type methods, the cost of solving subproblems exactly is quite expensive at each iteration step. As mentioned by Quiroz and Oliveira [20], for a computational implementation, it is important to analyze the convergence of the algorithm with an inexact iteration. Recently, Wang and López [31] proposed a modified proximal point algorithm on Hadamard manifolds, which extended the corresponding results of Xu [33] to Hadamard manifolds. The method consists of proximal step and Halpern's iteration, where the proximal step is an inexact one with the standard error criterion (1.3) (see Algorithm MP and its variant of [31]). As discussed above, when applying inexact proximal point algorithms for solving related problems, we prefer the relative error tolerance to the absolute one. However, to the best of our knowledge, we cannot find any inexact proximal point algorithm with a relative error tolerance on Riemannian manifolds.

Inspired and motivated by the research works above, in this paper, we extend an inexact proximal point algorithm with the relative error tolerance proposed by Han and He [10] from Euclidean spaces to Hadamard manifolds. Under suitable assumptions, we prove the sequence generated by the proposed method converges to the singularity of maximal monotone vector fields on Hadamard manifolds. Since the sequence generated by the proposed method is not Fejér (or quasi Fejér) convergent to the solution set of the problem, our techniques in this paper are mostly different with the previous ones for dealing with the Fejér (or quasi Fejér) convergent sequence on Hadamard manifolds. Moreover, we give an application to the optimization problem on Hadamard manifolds, which also generalizes and improves the corresponding results of Ferreira and Oliveira [8] and Li et al. [13].

## 2. Preliminaries

In this section, we recall some fundamental definitions, properties and notation of Riemannian manifolds, which can be found in any textbook on Riemannian geometry, for example, [24].

Let $\mathbb{M}$ be a connected $m$-dimensional manifold and let $x \in \mathbb{M}$. We always assume that $\mathbb{M}$ can be endowed with a Riemannian metric to become a Riemannian manifold. The tangent space of $\mathbb{M}$ at $x$ is denoted by $T_x\mathbb{M}$. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on $T_x\mathbb{M}$ with the associated norm $\| \cdot \|_x$, where the subscript $x$ is sometimes omitted. The tangent bundle of $\mathbb{M}$ is denoted by $T\mathbb{M} = \cup_{x \in \mathbb{M}} T_x\mathbb{M}$, which is naturally a manifold. Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{M}$ joining $x$ to $y$ (i.e. $\gamma(a) = x$ and $\gamma(b) = y$), we can define the length of $\gamma$ by $l(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then the Riemannian distance $d(x, y)$, which induces the original topology on $\mathbb{M}$, is defined by minimizing this length over the set of all such curves joining $x$ to $y$.

Let $\nabla$ be the Levi-Civita connection associated with the Riemannian metric. Let $\gamma$ be a smooth curve in $\mathbb{M}$. A vector field $X$ is said to be parallel along $\gamma$ iff $\nabla_{\gamma'} X = 0$. If $\gamma'$ itself is parallel along $\gamma$, we say that $\gamma$ is a geodesic (this notion is different from the corresponding one in the calculus of variations), and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, $\gamma$ is said to be normalized. A geodesic joining $x$ to $y$ in $\mathbb{M}$ is said to be minimal if its length equals $d(x, y)$.

A Riemannian manifold is complete if for any $x \in \mathbb{M}$ all geodesics emanating from $x$ are defined for all $-\infty < t < +\infty$. By the Hopf–Rinow Theorem, we know that if $\mathbb{M}$ is complete then any pair of points in $\mathbb{M}$ can be joined by a minimal geodesic. Moreover, $(\mathbb{M}, d)$ is a complete metric space and bounded closed subsets are compact.

We use $P_{\gamma,\cdot,\cdot}$ to denote the parallel transport on the tangent bundle $T\mathbb{M}$ along $\gamma$ with respect to $\nabla$, which is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(v) = V(\gamma(b)) \quad \text{for any } a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{M},$$

where $V$ is the unique vector field satisfying $\nabla_{\gamma'(t)} V = 0$ for all $t$ and $V(\gamma(a)) = v$. Then, for any $a, b \in \mathbb{R}$, $P_{\gamma,\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{M}$ to $T_{\gamma(b)}\mathbb{M}$. We will write $P_{y,x}$ instead of $P_{\gamma,y,x}$ in the case where $\gamma$ is a minimal geodesic joining $x$ to $y$; this will avoid any confusion.

Assuming that $\mathbb{M}$ be complete, the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x$ is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_x\mathbb{M}$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at $x$ with velocity $v$. Then $\exp_x tv = \gamma_v(t, x)$ for each real number $t$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for any $x \in \mathbb{M}$.

A complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Throughout the remainder of this paper, we will always assume that $\mathbb{M}$ is an $m$-dimensional Hadamard manifold. The following result is well known (see, for example, Theorem 4.1 of [24]).

**Proposition 2.1.** *Let $\mathbb{M}$ be a Hadamard manifold and $p \in \mathbb{M}$. Then $\exp_p : T_p\mathbb{M} \to \mathbb{M}$ is a diffeomorphism, and for any two points $p$, $q \in \mathbb{M}$, there exists a unique normalized geodesic joining $p$ to $q$, which is, in fact, a minimal geodesic.*

This proposition shows that $\mathbb{M}$ is diffeomorphic to the Euclidean space $\mathbb{R}^m$. Thus, we see that $\mathbb{M}$ has the same topology and differential structure as $\mathbb{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of the most important properties is described in the following proposition, which is taken from Proposition 4.5 of [24] and will be useful in our study. Recall that a geodesic triangle $\triangle(p_1p_2p_3)$ of a Riemannian manifold is a set consisting of three points $p_1$, $p_2$ and $p_3$, and three minimal geodesics joining these points.