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A completely positive representation of 0–1 linear programs with joint probabilistic constraints

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ABSTRACT

In this paper, we study 0–1 linear programs with joint probabilistic constraints. The constraint matrix vector rows are assumed to be independent, and the coefficients to be normally distributed. Our main results show that this non-convex problem can be approximated by a convex completely positive problem. Moreover, we show that the optimal values of the latter converge to the optimal values of the original problem. Examples randomly generated highlight the efficiency of our approach.

1. Introduction

In this paper, we study the following 0–1 linear program with joint probabilistic or chance constraints, called (LPJPC) hereafter:

$$\max\ c^T x$$

$$(LPJPC) \quad s.t.\ \boldsymbol{Pr}\{Tx \le D\} \ge 1-\alpha \tag{1}$$

$$w_t^T x = d_t, \quad t = 1, \ldots, m \tag{2}$$

$$\bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \tag{3}$$

$$x \in \{0, 1\}^n \tag{4}$$

where $c \in R^n$, $w_t \in R^n$, $\bar{w}_{\bar{t}} \in R^n$, $D = (D_1, \ldots, D_K) \in R^K$, $T = [T_1, \ldots, T_K]^T$ is a $K \times n$ random matrix, where T_k, $k = 1, \ldots, K$ is a random vector in R^n, and α is a prespecified confidence parameter. One of the fundamental problems is to solve LPJPC in the case where the number of the constraints under the probability is greater than one, i.e., $K > 1$.

Probabilistic constraint problems have been extensively studied in the literature for the last decades either in the case of individual probabilistic constraints; see for instance Charnes et al. [5], Prekopa [18,19], Kosuch and Lisser [12,13] or in the case of joint probabilistic constraints, cf. Miller and Wagner [15], Jagannathan [11], Henrion and Strugarek [10], Van Ackooij et al. [1], Cheng and Lisser [6], Luedtke et al. [14].

In this paper, we approximate the non-convex LPJPC with normally distributed matrix coefficients and independent matrix row vectors by a convex completely positive problem (CP for short). This naturally leads to semidefinite programming relaxations (called SDP hereafter) that are solvable in polynomial time and provide tight lower bounds. The cone approximation is performed by using the formulations proposed in [6].

The rest of this paper is organized as follows. In Section 2, we introduce the new CP formulation and its theoretical properties, and in Section 3 we present our computational experiment results to illustrate the strength of our CP formulation and the effectiveness of its SDP relaxation. Conclusions are given in the last section.

2. CP formulation

A symmetric matrix S is copositive if $y^T Sy \ge 0$ for all $y \ge 0$, and the set of all copositive matrices, denoted by C, is a closed, and convex cone (see for instance [3,8] for recent surveys).

The convex cone C of copositive matrices and its dual cone C^* of completely positive matrices have received great interest during the last two decades thanks to the diversity of CP formulations in different optimization topics, namely continuous and discrete optimizations. Several interesting NP-hard problems can be modeled as convex conic optimization problems over those cones (see for instance [3] and the references within). Our CP formulation relies on the approximations proposed in [6].

We assume that T_k, $k = 1, \ldots, K$ are independent multivariate normally distributed vectors with known mean vector $\mu_k = (\mu_{k1}, \ldots, \mu_{kn})$ and covariance matrix Σ_k.

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The normal-distribution assumption is motivated by its several theoretical characteristics amongst them the central limit theorem. From the practical point of view, several uncertainty sources could be w.l.o.g considered as normally distributed. However, independence of the random variables is important in our approach as our approximations are based on this assumption.

Ackooij et al. [1] derived an explicit way to calculate the gradient of the constraint functions for the LPJPC model with dependent random variables assumption. However, computing the gradients is highly demanding in terms of CPU time that it cannot be easily used within a B&B framework.

According to our assumptions, we can derive a deterministic reformulation of LPJPC as follows:

$$
\begin{aligned}
& \max\ c^T x \\
& s.t.\ \mu_k^T x + F^{-1}(p^{z_k})\|\Sigma_k^{1/2} x\| \le D_k, \quad k = 1,\dots,K \\
(P_0)\ & \sum_{k=1}^{K} z_k = 1, \quad z_k \ge 0, k = 1,\dots,K \\
& w_t^T x = d_t, \quad t = 1,\dots,m \\
& \bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \bar{t} = 1,\dots,\bar{m} \\
& x \in \{0,1\}^n
\end{aligned}
\tag{5}
$$

where $p = 1-\alpha$ and $F^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function F.

The problem (5) is equivalent to

$$
\begin{aligned}
& \max\ c^T x \\
& s.t.\ (F^{-1}(p^{z_k}))^2 x^T \Sigma_k x \le (D_k - \mu_k^T x)^2, \quad k = 1,\dots,K \\
& \mu_k^T x \le D_k, \quad k = 1,\dots,K \\
(P_1)\ & \sum_{k=1}^{K} z_k = 1, \quad z_k \ge 0, k = 1,\dots,K \\
& w_t^T x = d_t, \quad t = 1,\dots,m \\
& \bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \bar{t} = 1,\dots,\bar{m} \\
& x \in \{0,1\}^n.
\end{aligned}
$$

There are two steps to approximate the problem (5): first, we approximate $(F^{-1}(p^{z_k}))^2$ with a piecewise tangent approximation of z_k [6] and the approximation of LPJPC is an SOCP problem apart from the binary constraints. Second, we transform the SOCP constraints into linear constraints by using the linearization method.

Lemma 2.1. *With the piecewise tangent approximation of $(F^{-1}(p^{z_k}))^2$ and the linearization method, we have the approximation of (P_0) as follows:*

$$
\begin{aligned}
& OPT_{AP_0} = \max\ c^T x \\
& s.t.\ \langle \Sigma_k, Z^k \rangle \le D_k^2 - 2D_k\mu_k^T x + \langle \mu_k\mu_k^T, X\rangle, \quad k = 1,\dots,K \\
& \mu_k^T x \le D_k, \qquad \sum_{k=1}^{K} z_k = 1, \quad z_k \ge 0, k = 1,\dots,K \\
& Z_{i,j}^k \le U^+ X_{ij}, \qquad Z_{i,j}^k \le \hat{F}_k, \qquad Z_{i,j}^k \ge 0 \\
& Z_{i,j}^k \ge \hat{F}_k - (1 - X_{ij})U^+, \quad i,j = 1,\dots,n, k = 1,\dots,K \\
& \hat{F}_k \ge a_l + b_l z_k, \quad l = 0,1,\dots,N, \\
(AP_0)\ & X_{ij} \le x_i, \qquad X_{ij} \le x_j, \qquad X_{ij} \ge 0 \\
& X_{ij} \ge x_i + x_j - 1, \quad i,j = 1,\dots,n \\
& X_{ii} = x_i, \quad i = 1,\dots,n \\
& w_t^T x = d_t, \quad t = 1,\dots,m \\
& \bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \bar{t} = 1,\dots,\bar{m} \\
& x \in \{0,1\}^n
\end{aligned}
\tag{6}
$$

where $a_0 = 0, b_0 = 0$, U^+ is an upper bound of $\hat{F}_k$ and $\hat{F}_k = \max_{l=0,\dots,N}\{a_l + b_l \cdot z_k\}$ is a piecewise tangent approximation of $(F^{-1}(p^{z_k}))^2$. Moreover, the optimal value of (AP_0) is an upper bound of (P_0). Furthermore, $\lim_{N\to\inf} OPT_{AP_0} = OPT_{P_0}$.

Proof. First, we prove that $(F^{-1}(p^z))^2$ is convex on the interval $(0,1]$. The second derivative of $F^{-1}(p^z)$ is given by

$$
(F^{-1}(p^z))'' = \frac{(\ln p)^2 p^z [f(F^{-1}(p^z)) + F^{-1}(p^z)]}{[f(F^{-1}(p^z))]^2}.
$$

As $p \ge \frac{1}{2}$, then $F^{-1}(p^z)$ is nonnegative. Therefore, $(F^{-1}(p^z))''$ is nonnegative and $F^{-1}(p^z)$ is convex. Moreover, as the square function is non-decreasing and convex on the interval $[0,\infty)$, then $(F^{-1}(p^z))^2$ is convex. Second, by applying the standard linearization technique introduced in [9] and the theory presented in [6], we can show that the optimal value of (AP_0) is an upper bound of (P_0). Finally, applying the results of the piecewise linear approximation presented in [21] concludes the proof. □

By adding the slack variables $s_k, \hat{s}_k, \hat{s}_{kl}, \bar{s}_{\bar{t}}, Z^{k'}, Z^{k''}, Z^{k'''}, X'_{ij}, X''_{ij}, X'''_{ij}$, we get the standard formulation [4]:

$$
\begin{aligned}
& OPT_{AP_1} = \max\ c^T x \\
& s.t.\ \langle \Sigma_k, Z^k \rangle + 2D_k\mu_k^T x - \langle \mu_k\mu_k^T, X\rangle + s_k = D_k^2, \quad k = 1,\dots,K \\
& \mu_k^T x + \hat{s}_k = D_k, \qquad \sum_{k=1}^{K} z_k = 1, \quad k = 1,\dots,K \\
& U^+ X_{ij} - Z_{i,j}^k - Z_{i,j}^{k'} = 0, \qquad \hat{F}_k - Z_{i,j}^k - Z_{i,j}^{k''} = 0, \\
& \quad i,j = 1,\dots,n, k = 1,\dots,K \\
& Z_{i,j}^k - \hat{F}_k - X_{i,j}U^+ - Z_{i,j}^{k'''} = -U^+, \\
& \quad i,j = 1,\dots,n, k = 1,\dots,K \\
& \hat{F}_k - b_l z_k - \hat{s}_{kl} = a_l, \quad l = 1,\dots,N, k = 1,\dots,K \\
(AP_1)\ & x_i - X_{ij} - X'_{ij} = 0, \qquad x_j - X_{ij} - X''_{ij} = 0, \\
& \quad i,j = 1,\dots,n \\
& x_i + x_j - 1 - X_{ij} + X'''_{ij} = 0, \quad i,j = 1,\dots,n \\
& x_i - X_{ii} = 0, \quad i = 1,\dots,n \\
& w_t^T x = d_t, \quad t = 1,\dots,m \\
& \bar{w}_{\bar{t}}^T x + \bar{s}_{\bar{t}} = \bar{d}_{\bar{t}}, \quad \bar{t} = 1,\dots,\bar{m} \\
& z_k, s_k, \hat{s}_k, \bar{s}_{\bar{t}} \ge 0, \qquad \hat{s}_{kl} \ge 0, \qquad Z_{i,j}^k, Z_{i,j}^{k'}, Z_{i,j}^{k''}, Z_{i,j}^{k'''} \ge 0 \\
& X_{ij}, X'_{ij}, X''_{ij}, X'''_{ij} \ge 0, \quad i,j = 1,\dots,n \\
& x \in \{0,1\}^n.
\end{aligned}
\tag{7}
$$

For the sake of simplicity, problem (7) can be rewritten as follows:

$$
\begin{aligned}
& \max\ \hat{c}^T y \\
& s.t.\ \hat{w}_t y = \hat{d}_t, \\
& \quad t = 1,\dots,K(3n^2 + N + 2) + 3n^2 + n + m + \bar{m} + 1 \\
& y_i \in \{0,1\}, \quad i = 1,\dots,n \\
& y \in \mathbb{R}_+^{(4n^2+N+3)K+4n^2+n+\bar{m}}
\end{aligned}
\tag{8}
$$

where $\hat{c}$, $\hat{w}_t$ and $\hat{d}_t$ are defined accordingly.

Theorem 2.0.1. *(AP_1) is equivalent to the following completely positive problem:*

$$
\begin{aligned}
& OPT_{CP} = \max\ \hat{c}^T y \\
& s.t.\ \hat{w}_t y = \hat{d}_t,
\end{aligned}
$$