Operations Research Letters 41 (2013) 607–610

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Envy-free two-player *m*-cake and three-player two-cake divisions

Nicolas Lebert [a], Frédéric Meunier [b,*], Quentin Carbonneaux [c]

[a] Ecole des Ponts ParisTech, 6-8 avenue Blaise Pascal, Cité Descartes, 77455 Marne-la-Vallée cedex 2, France

[b] Université Paris Est, CERMICS, Ecole des Ponts ParisTech, 6-8 avenue Blaise Pascal, Cité Descartes, 77455 Marne-la-Vallée cedex 2, France

[c] Department of Computer Science, Yale University, New Haven, CT 06520, United States

ARTICLE INFO

Article history:
Received 7 May 2013
Received in revised form
25 July 2013
Accepted 31 July 2013
Available online 9 August 2013

Keywords:
Fair division
Envy-free
Matching
m-partite hypergraph
Polytope
Sperner's lemma

ABSTRACT

Cloutier, Nyman, and Su (2005) initiated the study of envy-free cake-cutting problems involving several cakes. They showed that when there are two players and two or three cakes it is possible to find envy-free cake-divisions requiring few cuts, under natural assumptions.

We prove that such a result also exists when there are two players and any number of cakes and when there are three players and two cakes. The proof relies on the fractional matching number in *m*-partite hypergraphs.

1. Introduction

1.1. Context

Cake-cutting problems ask whether it is possible to divide a cake among players in such a way that each of them believes the division is fair. These problems go back to Steinhaus [6] and have received a lot of attention. Many variations are possible depending on whether the pieces may be disconnected or not and on how "fair" is understood. In this note, we consider a division to be fair if all players consider their own pieces to be at least as valuable as any of the others. Dubins and Spanier [3] were able to prove that such a division exists provided that each player's preference is defined by a nonatomic measure over the cake. Unfortunately, a piece of cake in their result may be a collection of many (possibly infinite) disjoint connected subsets. Stromquist [7] improved their result by showing that such a division can be obtained by cutting the cake by $q-1$ planes, each parallel to a given plane, where q is the number of players. Moreover, he showed that such divisions exist for a larger class of "preferences". Informally, given a partition of the cake into pieces, we require simply that each player is able to say which pieces he prefers. We consider this kind of preferences in the present note and they are formally defined hereafter. According to Su [8], Forest Simmons found a constructive proof of Stromquist's theorem based on Sperner's lemma [5], the combinatorial counterpart of Brouwer's fixed point theorem.

In 2010, Cloutier, Nyman, and Su [1] asked whether extensions of this theorem are possible when there are more than one cake. Given m cakes, is it possible to divide each cake into a given number of connected pieces and assign one piece from each cake to each player in such a way that each player believes his assignment at least as good as any other assignment?

1.2. Model

Each cake is identified with the interval $[0, 1]$. A *division* of the cake i into r_i pieces is an r_i-tuple $\boldsymbol{x}_i = \left(x_{i1}, \ldots, x_{ir_i}\right)$, with $x_{ij} \geq 0$ for all $j \in [r_i]$ and $\sum_{j=1}^{r_i} x_{ij} = 1$, where x_{ij} is the size of the jth piece (ordered from left to right) of the cake i. Given a division $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ of the m cakes, a *piece selection* is the selection of one piece in each cake, i.e. it is an m-tuple $(j_1, \ldots, j_m) \in [r_1] \times \cdots \times [r_m]$. A player *prefers* a certain piece selection if that player does not think that any other piece selection is strictly better. For some divisions a player may be indifferent to two or more "preferred" piece selections.

We make the following assumptions on the preferences, which are the ones considered by Stromquist in the one-cake case.

* Corresponding author.
E-mail addresses: nicolas.lebert@ponts.org (N. Lebert), frederic.meunier@enpc.fr (F. Meunier), quentin.carbonneaux@yale.edu (Q. Carbonneaux).

0167-6377/$ – see front matter
http://dx.doi.org/10.1016/j.orl.2013.07.010

1. *Independence of preferences:* The preferences of one player do not depend of the choices made by the other players.
2. *The players are hungry:* A player will never choose an empty piece.
3. *Preference sets are closed:* If one player prefer the same piece selection for a convergent sequence of division, then that piece selection will be preferred at the limit.

A division of m cakes between q players is *envy-free* if there exist q disjoint piece selections and an assignment of the piece selections to the players such that each player prefers the piece selection he gets to any other piece selection among all possible piece selections for this division. The question studied by Cloutier, Nyman, and Su is whether there exists an integer $r(q, m)$, independent of the preferences, such that there exists an envy-free division of the m cakes with no more than $r(q, m)$ pieces per cake. Note that Stromquist's theorem asserts the existence of $r(q, 1)$ for any q and that $r(q, 1) = q$. Using a polytopal version of Sperner's lemma [2], Cloutier, Nyman, and Su proved the existence of $r(2, 2)$ and $r(2, 3)$ and that $r(2, 2) = 3$ and $r(2, 3) \leq 4$. They asked moreover whether $r(2, m) \leq m + 1$.

1.3. Main results

We contribute to the questions by proving the following theorem.

Theorem 1. *The integer $r(2, m)$ exists for any $m \geq 2$ and is such that $r(2, m) \leq m(m - 1) + 1$.*

In addition to the polytopal version of Sperner's lemma, the proof uses an inequality (Gyárfás's theorem) between the matching number and the fractional matching number in m-graphs.

We are also able to prove a first result involving three players.

Theorem 2. *The integer $r(3, 2)$ exists and is such that $r(3, 2) \leq 5$.*

1.4. Plan

In Section 2, we give the main tools used in the proofs, such as a polytopal version Sperner's lemma and Gyárfás's theorem. Section 3 is devoted to the proof of Theorem 1 and Section 4 to the proof of Theorem 2.

2. Tools

2.1. Sperner's labeling

Given a triangulation T of a polytope P, a *Sperner labeling* is a map $\lambda : V(\mathsf{T}) \to V(P)$, where $V(\mathsf{T})$ and $V(P)$ are respectively the vertex sets of T and P, such that $\lambda(v)$ is a vertex of the minimal face of P containing v. The following theorem is proved in [2]. Given a simplex σ, we denote its vertex set $V(\sigma)$. The polytopal version of Sperner's lemma already mentioned is the following theorem.

Theorem 3. *Let T be a triangulation of a polytope P. If λ is a Sperner labeling of T, then $\bigcup_{\sigma \in \mathsf{T}} \operatorname{conv}(\lambda(V(\sigma))) = P$.*

Note that this theorem implies that for any point x of P, there is a $\sigma \in \mathsf{T}$ with $\dim \sigma = \dim P$ such that $x \in \operatorname{conv}(\lambda(V(\sigma)))$ and such that $\operatorname{conv}(\lambda(V(\sigma)))$ is non-degenerate, i.e. of dimension $\dim P$.

2.2. Divisions, polytopes, owner-labeling, and preference-labeling

The divisions of the m cakes with r_i pieces in cake i are exactly the points of the polytope $P = \triangle_1 \times \cdots \times \triangle_m$, where $\triangle_i$ is the $(r_i - 1)$-simplex $\{(x_{i1}, \ldots, x_{ir_i}) \in \mathbb{R}_+^{r_i} : \sum_{j=1}^{r_i} x_{ij} = 1\}$. The polytope P – called the *polytope of divisions* – has $\Pi_{i=1}^m r_i$ vertices and is of dimension $\sum_{i=1}^m r_i - m$.

Following [1], we explain how to locate the envy-free divisions on P. We assume that a triangulation T of P is given. We label the vertices of T with an *owner-labeling*, which is a map $o : V(\mathsf{T}) \to [q]$ assigning a player to each vertex of the triangulation. We require moreover this owner-labeling to be *uniform*: on each simplex, the number of times each player appears as a label differs by at most one from any other player. In other words, given any simplex $\sigma \in \mathsf{T}$ and its vertex set $V(\sigma)$, we have $\big||o^{-1}(k) \cap V(\sigma)| - |o^{-1}(k') \cap V(\sigma)|\big| \leq 1$ for all $k, k' \in [q]$.

The following proposition is proved in [1]. The *mesh-size* of a triangulation is the maximum diameter of its simplices.

Proposition 1. *There exists a triangulation T admitting a uniform owner-labeling for any polytope, any number q of players, and of arbitrarily small mesh-size.*

Given a triangulation T of P with a uniform owner-labeling o, we define a new labeling $\lambda : V(\mathsf{T}) \to V(P)$ of the vertices of T with the vertices of P: the *preference-labeling*. Each vertex v of T is a point in P and as such corresponds to a division of the m cakes. The player $o(v)$ prefers some piece selection $(j_1, \ldots, j_m)$ for this division (in case of a tie, make an arbitrary choice). We define then $\lambda(v)$ to be the vertex of P with coordinates $(\lambda_{ij}(v))$ with $\lambda_{ij}(v) = 0$ except for the pairs (i, j_i) for which $\lambda_{ij_i}(v) = 1$. Since "the players are hungry", the map λ is a Sperner labeling.

2.3. m-graphs and matchings

A *hypergraph* is a pair $H = (V, E)$ where V is a finite set and E a family of subsets of V. The elements of V are called the *vertices* and the elements of E are called the *edges*. We denote by $\delta(v)$ the set of edges containing a vertex v. A *matching* is a collection of pairwise disjoint edges. A *fractional matching* is a vector $\boldsymbol{w} \in \mathbb{R}_+^E$ such that $\sum_{e \in \delta(v)} w_e \leq 1$ for all $v \in V$. Note that a matching is a fractional matching with $w_e \in \{0, 1\}$ for all edges $e \in E$. The maximum cardinality of a matching, called the *matching number*, is denoted $\nu(H)$ and the maximum possible value of $\sum_{e \in E} w_e$ for a fractional matching $\boldsymbol{w}$, called the *fractional matching number*, is denoted $\nu^*(H)$. Since a matching is a fractional matching, we have $\nu^*(H) \geq \nu(H)$.

An *m-partite hypergraph*, or *m-graph*, is a hypergraph $H = (V, E)$ whose vertex set is the disjoint union of m sets $V_1, \ldots, V_m$ and such that each edge intersects each V_i in exactly one vertex. The following theorem, proved by Gyárfás (according to Füredi [4]), shows that the gap between the fractional matching number and the matching number is not too large for m-graphs.

Theorem 4 (*Gyárfás's Theorem*)**.** *If H is an m-graph, then $\nu^*(H) \leq (m - 1)\nu(H)$.*

2.4. A hypergraphic condition of existence of disjoint envy-free piece selections

We consider a triangulation T of the polytope of division P and assume that we have for T an owner-labeling o and a preference-labeling λ. As in [1], we will use Theorem 3 to get the existence of a simplex $\sigma \in \mathsf{T}$ with special features to obtain eventually the existence of an envy-free division of the cakes with no more than r_i pieces per cake i. However, we introduce an additional combinatorial criterion based on some hypergraph properties in an attempt to systematize the reasoning.

Given a simplex $\sigma \in \mathsf{T}$, we define $H(\sigma)$ to be the m-graph with $V_i = \{(i, j) : j \in [r_i]\}$ for $i = 1, \ldots, m$. The edges of $H(\sigma)$ are