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Disjoint cycles with different length in 4-arc-dominated digraphs

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ABSTRACT

A d-arc-dominated digraph is a digraph D of minimum out-degree d such that for every arc (x, y) of D, there exists a vertex u of D of out-degree d such that (u, x) and (u, y) are arcs of D. Henning and Yeo [Vertex disjoint cycles of different length in digraphs, SIAM J. Discrete Math. 26 (2012) 687–694] conjectured that a digraph with minimum out-degree at least four contains two vertex-disjoint cycles of different length. In this paper, we verify this conjecture for 4-arc-dominated digraphs.

1. Introduction

Our notations mainly follow that of Bang-Jensen and Gutin [3]. In a digraph, a cycle of length one is a loop and a cycle of length three is called a triangle. All digraphs contained in this paper can have loops and cycles of length two but no parallel arcs. A digraph without cycles of length at most two is called an oriented digraph, and a digraph without loops and parallel arcs is called a strict digraph.

Let $D = (V(D), A(D))$ denote a digraph, its order is $|V(D)|$. Let $x, y \in V(D)$, if there is an arc from x to y, then we write $x \to y$ and say x dominates y. Given a subset X of $V(D)$, the sub-digraph of D induced by X is the digraph $D[X] := (X, A')$, where A' is the set of all arcs in $A(D)$ that start and end in X. Two sub-digraphs D_1 and D_2 of D are disjoint if their vertex sets are. If X and Y are two disjoint subsets of $V(D)$ or sub-digraphs of D such that every vertex of X dominates every vertex of Y, then we say that X dominates Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the property that there is no arc from Y to X. If the set X is composed of only one vertex v we simply say that v dominates Y. The set Y is dominated if there exists a vertex dominating it. The set X dominates a sub-digraph D' of D if it dominates its vertex set $V(D')$. We use $a^+(X, Y)$ to denote the number of arcs from X to Y, and $a^-(X, Y)$ denote the number of arcs from Y to X.

For every vertex $v \in V(D)$, let $N_D^+(v) := \{u \in V(D) | v \to u\}$ be the out-neighborhood of v in D, namely, the set of vertices dominated by x in D, and let $d_D^+(v) = |N_D^+(v)|$ be the out-degree of v in D. Similarly, the in-neighborhood of x in D is denoted by $N_D^-(v)$, which is the set of vertices dominating v in D, and let $d_D^-(v) = |N_D^-(v)|$ be the in-degree of v in D. The minimum out-degree and the minimum in-degree of D are defined by $\delta^+(D) = \min\{d_D^+(v) : v \in V(D)\}$ and $\delta^-(D) = \min\{d_D^-(v) : v \in V(D)\}$, respectively. A digraph D is k-regular if, for any $x \in V(D)$, $d_D^+(x) = d_D^-(x) = k$. A path or a cycle of D always means a directed path or a directed cycle of D. If $C = x_1x_2x_3 \dots x_rx_1$ is a cycle in D, then $C[x_i, x_j]$ denotes the path $x_ix_{i+1} \dots x_j$ along the direction of C, where all indices are taken modulo r. In particular, if $i = j$, then $C[x_i, x_j]$ denotes the empty path with vertex x_i. A d-arc-dominated digraph is a digraph D of minimum out-degree d such that for every arc (x, y) of D, there exists a vertex u of D of out-degree exactly d such that (u, x) and (u, y) are arcs of D.

A tournament T is a digraph T such that for any two distinct vertices x and y, exactly one of the couples $x \to y$ and $y \to x$ is an arc of T. The following conjecture, due to Bermond and Thomassen [4], gives a relation between the minimum out-degree and the maximum number of disjoint cycles in a digraph.

Conjecture 1.1 ([4]). *Let $k \geq 1$ be an integer, any digraph D with $\delta^+(D) \geq 2k - 1$ contains k disjoint cycles.*

Conjecture 1.1 is trivial for $k = 1$. Thomassen [11] verified the case when $k = 2$ by a nice induction technique. Lichiardopol et al. [9] proved the case when $k = 3$. Note that Alon [1] proved that a lower bound of $64k$ on the minimum out-degree gives k disjoint cycles. Along a different line, it was shown in [5] that every tournament with both minimum out-degree and minimum in-degree at least $2k - 1$ contains k disjoint triangles. Recently, Bang-Jensen et al. [2] verified Conjecture 1.1 for tournament. In the proofs of Thomassen [11] and Lichiardopol et al. [9], a crucial role is played by an oriented 2-arc-dominated digraph and an oriented 3-arc-dominated digraph, respectively. In general, Lichiardopol posed

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the problem (see Problem 912 (BB20.4) in [6]): characterize d-arc-dominated digraphs for any positive integer d.

Lichiardopol [6] also posed the following conjecture there, which could be viewed as an important step to attack Conjecture 1.1.

Conjecture 1.2. *A d-arc-dominated digraph with $d \geq 2k-1$ contains k disjoint cycles.*

N. D. Tan [10] answered Lichiardopol's problem [6] for the case $d = 3$, and he showed that an oriented digraph is 3-arc-dominated if each of its connected components is isomorphic to two known exceptional graphs. These two exceptional graphs (see [10]) always have two disjoint cycles with the same length. As noted in [8], there are examples of 3-regular digraphs where all pairs of vertex disjoint cycles have the same length. Henning and Yeo [8] proved that all 4-regular digraphs have two disjoint cycles of different length, and also proposed the following conjecture.

Conjecture 1.3 (*[8]*)**.** *Let D be a digraph. If $\delta^+(D) \geq 4$, then D contains two disjoint cycles of different length.*

Motivated by this conjecture and the main result of Bang-Jensen et al. [2], we [7] show that Conjecture 1.3 is true for tournament.

Theorem 1.4 (*[7]*)**.** *Let T be tournament with $\delta^+(T) \geq 3$, then T contains a cycle of length three and a cycle of length four, such that these two cycles are disjoint, unless T is isomorphic to some known graphs.*

In this paper, we prove Conjecture 1.3 is true for any 4-arc-dominated digraph.

Theorem 1.5. *Let D be a 4-arc-dominated digraph, then D contains two disjoint cycles with different length.*

2. Proof of Theorem 1.5

We proceed by contradiction. Suppose that the statement of Theorem 1.5 is false and consider a counter-example with the minimum number of vertices. Let D be a counter-example to the statement of Theorem 1.5 with the smallest of number of vertices, and subject to this with the smallest number of arcs. Then every vertex in $V(D)$ has out-degree exactly four. Suppose that there exists a vertex u of D with out-degree at least 5. Let $u \to v \in A(D)$. Then $D-(u, v)$ is a digraph of minimum out-degree 4. For arbitrary arc (x, y) of $D-(u, v)$, there exists a vertex w of minimum out-degree 4 in D, such that (w, x) and (w, y) are arcs of D. But then (w, x) and (w, y) are arcs of $D-(u, v)$, because of $w \neq u$. So, (x, y) is 4-arc-dominated in $D-(u, v)$, which implies that $D-(u, v)$ is a 4-arc-dominated digraph. Clearly $D-(u, v)$ is a counter-example for Theorem 1.5, which by minimality of the size of a counter-example, is not possible. So, every vertex of D has out-degree 4. We begin with an easy observation and then establish some fundamental properties of D.

Lemma 2.1. *If D is a strict digraph and $\max\{\delta^+(D), \delta^-(D)\} = k > 0$, then D contains a directed cycle of length at least $k+1$.*

Lemma 2.2. *The following hold.*

(i) *The digraph D is an oriented digraph.*
(ii) *The in-neighborhood of every vertex in D contains a cycle of length at least three.*
(iii) *The digraph D contains a triangle.*

Proof. (i) Suppose that C is a cycle of D with length at most two. If C is a loop, the digraph obtained from D by removing the vertex of C has minimum out-degree at least three, thus contains a cycle C' with length at least two, a contradiction. Hence, we may assume that D is strict and C is a cycle of length two, note that the induced sub-digraph D' of D obtained by removing the vertices of C has minimum out-degree at least two, so D' contains a cycle of length at least three by Lemma 2.1, which disjoints with C, a contradiction.

(ii) By the minimality of D, each vertex $x \in V(D)$ satisfies $d_D^-(x) \geq 1$. Choose any one in-neighbor of x, say v. Note that $v \to x$ is dominated, that is, there exists $y \in V(D)$ with $y \to x$ and $y \to v$. Therefore, the digraph $D[N_D^-(x)]$ has in-degree at least one and thus contains a cycle. Combining with (i), we complete the proof of (ii).

(iii) Suppose that D contains no triangle. This implies that for each $v \in V(D)$, $d_D^-(v) \geq 4$ by (i) and (ii). We claim that D is 4-regular; otherwise,

$$4|V(D)| < \sum_{x \in V(D)} d_D^-(x) = \sum_{x \in V(D)} d_D^+(x) = 4|V(D)|, \tag{1}$$

a contradiction. Then, by the theorem of Henning and Yeo [8], D contains two disjoint cycles of different length, which contradicts the fact that D is a counter-example. □

We need the following lemma which was discovered by Lichiardopol et al. [9]. This lemma plays a very important role in our proof.

Lemma 2.3 (*[9]*)**.** *Let D be an arc-dominated oriented digraph, and let $X \subset V(D)$ such that $D[X]$ is either acyclic or an induced cycle of D. Then there exists a cycle C disjoint from $D[X]$ such that every vertex of C has at least one out-neighbor in X.*

Definition 2.4. Let T_1 and T_2 denote two disjoint triangles in D, such that each vertex of $V(T_2)$ has at least one out-neighbor in $V(T_1)$ and $a^+(V(T_1), V(T_2)) > 0$, then we say that T_1 and T_2 are two *good triangles*, and denoted by $\overrightarrow{T_1 \Leftarrow T_2}$.

Lemma 2.5. *Let $\overrightarrow{T_1 \Leftarrow T_2}$. Then $D[V(T_1 \cup T_2)]$ contains two cycles of different length (not necessarily disjoint).*

Proof. Let $T_1 = x_1y_1z_1x_1$ and $T_2 = x_2y_2z_2x_2$. Since $a^+(V(T_1), V(T_2)) > 0$, without loss of generality, we may suppose that $x_1 \to x_2$. Then $y_1z_1x_1x_2y_2z_2$ is a Hamiltonian path P of $D[V(T_1 \cup T_2)]$. Since each vertex of T_2 has at least 2 out-neighbors in $V(T_1 \cup T_2)$, z_2 has at least two out-neighbors in P, which yields two cycles of different length. This proves Lemma 2.5. □

We continue the proof. By Lemma 2.1, D is arc-dominated oriented graph and contains a triangle, denoted by C_1. Furthermore, note that $|V(D)| \geq 9$.

Claim 2.1. *D does not contain two good triangles.*

Proof. Suppose not. D contains two good triangles, say $T_1 = x_1y_1z_1x_1$ and $T_2 = x_2y_2z_2x_2$ and $\overrightarrow{T_1 \Leftarrow T_2}$. By Lemma 2.5 and the fact that D is a counter-example, the digraph D' obtained by removing $V(T_1 \cup T_2)$ from D is acyclic. This implies that there exists a vertex $u \in V(D')$ having no out-neighbor in $V(D')$, so u has exactly four out-neighbor in $V(T_1 \cup T_2)$. Furthermore, by Lemma 2.2, $D[N_D^-(u)]$ contains a cycle with length at least three, say C'. Clearly, $V(C') \cap V(T_1 \cup T_2) \neq \emptyset$, as otherwise, by Lemma 2.5, D contains two disjoint cycles of different length, a contradiction. We consider two cases.

Case 1. *One of $V(T_1)$ and $V(T_2)$ belongs to $N_D^+(u)$.*

Without loss of generality, say $V(T_1) \cup \{y_2\} \subseteq N_D^+(u)$. Then $u \not\to x_2$, as otherwise, $uy_2z_2x_2u$ is a cycle of length four, which disjoints from T_1, a contradiction. By Lemma 2.2(ii), $z_2 \in V(C')$ and so $z_2 \to u$. Let $a \in V(D' \cap C')$ such that $z_2 \to a$. Then $z_2auy_2z_2$ is a cycle of length four, which disjoints from T_1, a contradiction. This completes the proof of Case 1.

Case 2. *$|N_D^+(u) \cap V(T_1)| = 2$ and $|N_D^+(u) \cap V(T_2)| = 2$.*

Without loss of generality, we may assume that $\{y_1, z_1, y_2, z_2\} = N_D^+(u)$. By Lemma 2.2, one of x_1 and x_2 belongs to $V(C')$. Without loss of generality, we may suppose that $x_1 \in V(C')$. Then $x_1uy_1z_1x_1$ and T_2 are two disjoint cycles of different length, a contradiction. This completes the proof of Case 2. □