Operations Research Letters 41 (2013) 679–684

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Binary group facets with complete support and non-binary coefficients

Yan Shu [a], Sunil Chopra [b], Ellis L. Johnson [c], Sangho Shim [b,*]

[a] *Sabre Holdings Corporation, 3150 Sabre Drive, Southlake, TX 76092, United States*

[b] *J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208, United States*

[c] *School of Industrial and Systems Engineering, Georgia Institute of Technology, 765 Ferst Drive NW, Atlanta, GA 30332, United States*

ARTICLE INFO

Article history:
Received 20 December 2012
Received in revised form
20 September 2013
Accepted 26 September 2013
Available online 5 October 2013

Keywords:
Group problems
Binary groups
Lifting

ABSTRACT

We identify binary group facets with complete support and non-binary coefficients. These inequalities can be used to obtain new facets for larger problems using Gomory's homomorphic lifting.

1. Group polyhedra and homomorphic lifting

Most known binary group facets [3,4] are binary (or have only (0:1)-coefficients) with an incomplete support; i.e., not all of the coefficients are non-zero. In this paper, we identify families of non-binary facets with complete support and show that together with known (0:1)-facets and lifted facets, these inequalities completely describe the binary group polyhedron in low dimension.

The group problem was introduced by Gomory [4], and he gave some results specific to binary and ternary groups. Here, we only consider binary groups: finite abelian groups each element of which has order two. It can be shown that such groups are isomorphic to the direct product of cyclic groups of order two, denoted as C_2. The direct product C_2^n is isomorphic to the group of 0–1 vectors with n elements with the group operation being addition modulo 2. For applications of binary groups, we refer to Gastou and Johnson [3] and Johnson [7].

A *homomorphism* is a mapping $\varphi : G \to \bar{G}$, from group G onto $\bar{G}$, which satisfies

$$\varphi(g+h) = \varphi(g) + \varphi(h) \quad \text{for } \forall g, h \in G.$$

The *kernel* of φ is the subset K of G, where

$$K = \{g \in G : \varphi(g) = \bar{0}\},$$

where $\bar{0}$ is the zero of $\bar{G}$. It is well-known that K is a subgroup of G. For every element $g \in G$, the kernel K defines a *coset* $g + K$, where

$$g + K = \{h : h = g + k \text{ for } k \in K\}.$$

The image $\bar{G}$ of φ is isomorphic to the factor group G/K of cosets, and every element of G in a coset $g + K$ maps onto the same element of $\bar{G}$.

Given a group G, let $M \subseteq G \backslash \{0\}$ and $b \in M$. Let M be a nonempty subset of a group G without 0 and let b be a non-zero element of G. The *group problem* is to minimize $\sum_{g\in M} c_g t_g$ subject to

$$\sum_{g\in M} g t_g = b,$$

where the variables t_g are nonnegative integers and every element c_g of the objective function is nonnegative. The *group polyhedron* $P(G, M, b)$ is the convex hull of the solution vectors $(t_g : g \in M)$ to the group problem. The group polyhedron is known in [4] to be full-dimensional and the facets are uniquely represented by inequalities with the right-hand side 1. If the underlying group G is binary, $P(G, M, b)$ is a *binary* group polyhedron. If $M = G \setminus \{0\}$, we call $P(G, M, b)$ the *master* group polyhedron and denote it simply by $P(G, b)$. In his work on group polyhedra and master group polyhedra, Gomory defined families of facets of such polyhedra when the underlying group is cyclic or a binary group. For example, Gomory and Johnson defined two-slope facets for master cyclic group polyhedra in [5] and [6]. Basu, Hildebrand, Köppe and Molinaro [1] and Cornuéjols and Molinaro [2] have defined other families of facets for such polyhedra, including (three or more)-slope theorems.

* Corresponding author.
E-mail address: shim@northwestern.edu (S. Shim).

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http://dx.doi.org/10.1016/j.orl.2013.09.009

Gomory [4] showed that for any master group polyhedron $P(G, b)$, any homomorphism φ of G onto the group H with $\varphi(b) \neq 0$ and any facet $\bar{\pi}$ of $P(H, \varphi(b))$, one obtains a facet π for $P(G, b)$ by taking

$$\pi(g) = \begin{cases} \bar{\pi}(\varphi(g)) & \text{if } \varphi(g) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

He refers to this facet π as obtained by *lifting up* $\bar{\pi}$ from H to G via the homomorphism φ. We will refer to any facet obtained using this procedure as a *homomorphic lifting*. He also gives a converse theorem: every facet π of G with any coefficient $\pi_g = 0$ comes from a homomorphic lifting where the kernel $K = \{k \in G \setminus \{0\} : \pi(k) = 0\}$ and $b \notin K$. In other words, large facets with zero coefficients of a master group polyhedron can be obtained from smaller facets with no zero coefficient (complete support) of master group polyhedra of factor groups. In this paper, we identify families of non-binary facets with complete support for the master binary group polyhedron. We assume that $b = \mathbf{1}$, where $\mathbf{1}$ is a vector of all 1's. The binary group facets of $P(G, b)$ with $b \neq \mathbf{1}$ can be induced [4] from those with $b = \mathbf{1}$ by *automorphism*; *i.e.*, one-to-one homomorphism.

2. Binary group facets with complete support

For a master group polyhedron $P(G, b)$, the nonnegativity constraints are known to be facets and will be referred to as the *trivial* facets. All nontrivial facets will be referred to as *group facets* (or *binary group facets* if G is binary). Although there are exponentially many group facets (see [6]) for the master group problem, Gomory [4] has shown that all group facets can be characterized using polynomially many subadditive inequalities:

Theorem 1 (*Gomory [4]*). *The coefficient vectors $\pi \in \mathbb{R}^{G\setminus\{0\}}$ of the group facets $\pi t \geq \pi_b > 0$ are the extreme rays to the following system describing the subadditive cone $S(G, b)$; for all $g, g' \in G \setminus \{0, b\}$ with $g + g' \neq 0$,*

1. $\pi(g) + \pi(b - g) = \pi_b$, *(complementarities)*
2. $\pi(g) + \pi(g') \geq \pi(g + g')$, *(subadditivities)*
3. $\pi \geq 0$.

Note that $b - g = b + g$ (modulo 2) in the complementarities because the "$-$" and "$+$" operation are equivalent modulo 2. We define the *group facet polytope* $\Pi(G, b)$ to be the convex hull of the group facets π with $\pi_b = 1$. In other words, $\Pi(G, b)$ is the intersection of the subadditive cone $S(G, b)$ with $\pi_b = 1$.

Consider a subgroup H of a binary group G of *index* 2, *i.e.*, $|G| = 2|H|$. Gomory [4] has shown that if $b \notin H$,

$$\sum_{g \notin H} t_g \geq 1 \tag{1}$$

is a group facet that is referred to as a *binary facet* or a $(0 : 1)$-facet. We next identify some facets with non-binary coefficients.

2.1. $(\frac{1}{3} : \frac{2}{3})$-facets

For a subgroup H of a binary group G of index 2, define π^H, where $\pi^H_g = 2/3$ for $g \in H$ and $\pi^H_g = 1/3$ for $g \in G \setminus H$. We now show that π^H is a group facet of $P(G, b)$ by proving that it is an extreme point of $\Pi(G, b)$.

Theorem 2. *Let $n \geq 4$. If H is a subgroup of the binary group $G = C_2^n$ of index 2 and $b \in G \setminus H$, then $\pi^H = (\pi^H_g)_{g \in G\setminus\{0\}}$ is a group facet of $P(G, b)$ where*

$$\begin{aligned} \pi^H_g &= 1/3 \quad \text{for } g \in G \setminus (H \cup \{b\}), \\ &= 2/3 \quad \text{for } g \in H \setminus \{0\}, \\ &= 1 \quad \text{for } g = b. \end{aligned}$$

Proof. We can easily see that π^H is not a facet of $P(G, b)$ when $n \leq 3$. Assume $n \geq 4$. The nonnegative vector π^H defined in the theorem satisfies the complementarities and the subadditivities in Theorem 1: The complementary element of g with respect to b is $b - g = b + g$ because the inverse of g is itself. The subgroup H contains either g or its complementary element $b + g$, and $H + b$ contains the other. The complementarities follow that one of π^H_g and π^H_{g+b} is $1/3$ and the other is $2/3$. The subadditivities hold true, as $\pi^H_g + \pi^H_h = 1/3 + 1/3 < \pi^H_{g+h} = 1$ would contradict the complementarity.

In order to show that π^H is a group facet, we show that π^H is uniquely determined by its binding constraints among the system in Theorem 1. Consider the following binding constraints,

$$\pi_{h_1+b} + \pi_{h_2+b} = \pi_{h_1+h_2}, \tag{2}$$

where $h_1, h_2 \in H \setminus \{0\}$ and $h_1 \neq h_2$.

Assume that a vector π' in $\mathbb{R}^{G\setminus\{0\}}$ satisfies the complementarities and the equations in (2). Then, for every non-zero $h_1 \in H$,

$$\begin{aligned} \sum_{h \in H\setminus\{0,h_1\}} \pi'_h &= \sum_{h \in H\setminus\{0,h_1\}} \pi'_{h+h_1} \\ &= \sum_{h \in H\setminus\{0,h_1\}} \left(\pi'_{h_1+b} + \pi'_{h+b}\right) \\ &= (|H| - 2)\pi'_{h_1+b} + \sum_{h \in H\setminus\{0,h_1\}} \pi'_{h+b} \\ &= (|H| - 3)\pi'_{h_1+b} + \sum_{h \in H\setminus\{0\}} \pi'_{h+b}, \end{aligned}$$

which implies

$$\begin{aligned} (|H| - 4)\pi'_{h_1+b} &= -\pi'_{h_1+b} + (|H| - 3)\pi'_{h_1+b} \\ &= -\pi'_{h_1+b} + \sum_{h \in H\setminus\{0,h_1\}} \pi'_h - \sum_{h \in H\setminus\{0\}} \pi'_{h+b} \\ &= -\pi'_b + \pi'_b - \pi'_{h_1+b} + \sum_{h \in H\setminus\{0,h_1\}} \pi'_h - \sum_{h \in H\setminus\{0\}} \pi'_{h+b} \\ &= -\pi'_b + (\pi'_{h_1} + \pi'_{h_1+b}) - \pi'_{h_1+b} \\ &\quad + \sum_{h \in H\setminus\{0,h_1\}} \pi'_h - \sum_{h \in H\setminus\{0\}} \pi'_{h+b} \\ &= -\pi'_b + \pi'_{h_1} + \sum_{h \in H\setminus\{0,h_1\}} \pi'_h - \sum_{h \in H\setminus\{0\}} \pi'_{h+b} \\ &= -1 + \sum_{h \in H\setminus\{0\}} \pi'_h - \sum_{h \in H\setminus\{0\}} \pi'_{h+b}. \end{aligned} \tag{3}$$

Since the right-hand side (3) is constant for all $h_1 \in H$, the components π'_{h+b} are constant ($= \beta$) over all non-zero $h \in H$. The components π'_h are shown to be constant ($= \alpha$) over all non-zero $h \in H$ by using the complementarities. Eq. (2) implies that $2\beta = \alpha$. The result thus follows. □

The proof of Theorem 2 is instructive and the proof technique can be used for the remaining theorems among which Theorem 5 is most complicated and is proved in Section 3. We generalize Theorem 2 to a subgroup H of index 4 ($|G| = 4|H|$) as follows:

Theorem 3. *For $n \geq 5$, let H be a subgroup of the binary group $G = C_2^n$ of index 4 and $b \in G \setminus H$. If $a \in G \setminus (H \cup (b + H))$, we have the factor group*

$$G/H = \{H, a + H, a + b + H, b + H\}.$$

Let (S, T) be a nonempty partition of H with $|S| \geq 3$ and $|T| \geq 3$. If

$$U = (a + S) \cup (a + b + T) \cup ((b + H) \setminus \{b\}),$$