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On allocation of redundancies in two-component series systems

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ABSTRACT

Allocation of active [standby] redundancies in a system is a topic of great interest in reliability engineering and system safety because optimal configurations can significantly increase the reliability of a system. In this paper, we study the problem of allocating two exponentially distributed active [standby] redundancies in a two-component series system using the tools of stochastic ordering. We establish two interesting results on likelihood ratio ordering which have no restriction on the parameters.

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1. Introduction

In reliability engineering, system safety is always one of the main concerns, especially for systems that require high reliability such as nuclear reactors and power supply systems in hospitals. It has been shown that a redundancy allocation technique can be used to improve the reliability of the system. In this regard, it is of great interest to allocate redundant component(s) in a system with the aim of optimizing the lifetime of the resulting system in reliability engineering and system safety.

The topic of how to allocate redundant components in a system to enhance the system reliability has been studied extensively. This topic generates a lot of interesting theoretical results with numerous practical applications in reliability engineering and system safety; see, for example, [1–11,13–17,19,18].

There are two commonly used types of redundancy—the active redundancy and the standby redundancy. For active redundancy, available spares are put in parallel to components of the system and these spares start functioning simultaneously as original components. For standby redundancy, spares are attached to components of the system in a way that a spare starts functioning right after the component to which it is attached failed. For these two different types of redundancies, the performance of different allocations can be measured by stochastic comparisons between the lifetimes of resulting systems in the sense of various stochastic orders. For some recent results on stochastic comparisons in series and parallel systems with redundancy, one can refer to [6,15,19].

In this paper, we will establish two results of comparison in likelihood ratio ordering for allocating two active and two standby redundancies to a series system with two nodes under the exponential framework, respectively. In Section 2, we present the main results (Theorems 1 and 2). The theoretical proofs of these main results are presented in Section 3.

2. Main results

We first define the notation and terminology in redundancy allocation that are used in the paper. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*. Let X and Y be two random variables with common support $\Re_+ = [0, \infty)$, density functions f_X and f_Y, distribution functions F_X and F_Y, respectively. Then, $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ are the survival functions of X and Y, respectively. Denote by $h_X = f_X/\overline{F}_X$ and $h_Y = f_Y/\overline{F}_Y$ the hazard rate functions of X and Y, and $r_X = f_X/F_X$ and $r_Y = f_Y/F_Y$ the reversed hazard rate functions of X and Y, respectively. X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \Re_+$; X is said to be smaller than Y

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in the hazard rate order (denoted by $X \le_{hr} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in $x \in \Re_+$; or $h_X(x) \ge h_Y(x)$ for all $x \in \Re_+$; X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \le_{rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in \Re_+$; or $r_X(x) \le r_Y(x)$ for all $x \in \Re_+$; X is said to be smaller than Y in the likelihood ratio order (denoted by $X \le_{lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \Re_+$; X is said to be smaller than Y in the increasing concave order (denoted by $X \le_{icv} Y$) if $\int_0^x \mathrm{P}(Y > t)\,\mathrm{d}t \ge \int_0^x \mathrm{P}(Y > t)\,\mathrm{d}t$ for all $x \in \Re_+$; X is said to be smaller than Y in the stochastic precedence order (denoted by $X \le_{sp} Y$) if $\mathrm{P}(X > Y) \le \mathrm{P}(Y > X)$. For a comprehensive discussion on various stochastic orders, one may refer to [12].

Let X_1, X_2, Y_1 and Y_2 be independent random variables denoting the lifetimes of the components C_1, C_2, and the redundancies R_1 and R_2, respectively. Suppose that S is a series system with components C_1 and C_2, and we are interested in the allocation of the redundancies R_1 and R_2 in order to obtain the optimal configuration of the resulting system. Specifically, in the active redundancy case, we want to compare the lifetimes

$$J_1 = \min(\max(X_1, Y_1), \max(X_2, Y_2))$$

and

$$J_2 = \min(\max(X_1, Y_2), \max(X_2, Y_1));$$

and in the standby redundancy case, we want to compare the lifetimes

$$Z_1 = \min(X_1 + Y_1, X_2 + Y_2) \quad \text{and}$$
$$Z_2 = \min(X_1 + Y_2, X_2 + Y_1).$$

In the rest of the paper, we focus on discussing the case when $X_i =_{st} Y_i,\ i = 1, 2$, that is, we assume that each redundancy has the same distribution as one of the components of the series system. In this case, [17] proved that if $X_1 \le_{hr} X_2$ and if the ratio of hazard rate functions of X_2 and X_1 is decreasing, then $J_1 \le_{hr} J_2$, while [2] proved that if $X_1 \le_{rh} X_2$ and if the ratio of reversed hazard rate functions of X_1 and X_2 is increasing, then $J_1 \le_{rh} J_2$. Let F_1 $[F_2]$, $\overline{F}_1$ $[\overline{F}_2]$, h_1 $[h_2]$ and r_1 $[r_2]$ be the distribution, survival, hazard rate and reversed hazard rate functions of X_1 $[X_2]$, respectively. Recently, [8] further improved the result of [17] and proved that if $X_1 \le_{st} X_2$ and $h_1(t)F_2(t) \ge h_2(t)F_1(t)$, then $J_1 \le_{hr} J_2$. They also obtained the result on reversed hazard rate order, i.e., if $X_1 \le_{st} X_2$ and $r_2(t)\overline{F}_1(t) \ge r_1(t)\overline{F}_2(t)$, then $J_1 \le_{rh} J_2$. For the standby case, [9] proved that if $X_1 \le_{icv} X_2$ and $Y_1 \le_{st} Y_2$, and X_1 or X_2 has a convex survival function on $[0, +\infty)$, then $Z_1 \le_{sp} Z_2$.

The following theorems are the main results of comparison in terms of the likelihood ratio ordering for allocating two active and two standby redundancies to a series system with two nodes under the exponential framework.

Theorem 1. *Let X_1, X_2, Y_1 and Y_2 be independent exponential random variables with rate parameters $\lambda_1, \lambda_2, \lambda_1$ and λ_2, respectively. Then,*

$$J_1 \le_{lr} J_2.$$

Theorem 2. *Let X_1, X_2, Y_1 and Y_2 be independent exponential random variables with rate parameters $\lambda_1, \lambda_2, \lambda_1$ and λ_2, respectively. Then,*

$$Z_1 \le_{lr} Z_2.$$

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. The case when $\lambda_1 = \lambda_2$ is trivially true. It can be observed that the case when $\lambda_1 > \lambda_2$ is actually equivalent to the case when $\lambda_1 < \lambda_2$ and, hence we consider that $\lambda_1 > \lambda_2$ in the following. The survival function and the probability density function of J_1 can be written as

$$\overline{F}_{J_1}(x) = 4e^{-(\lambda_1+\lambda_2)x} - 2e^{-(2\lambda_1+\lambda_2)x} - 2e^{-(\lambda_1+2\lambda_2)x} + e^{-2(\lambda_1+\lambda_2)x}$$

and

$$\begin{aligned} f_{J_1}(x) = 2\big[&2(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x} - (2\lambda_1+\lambda_2)e^{-(2\lambda_1+\lambda_2)x} \\ &- (\lambda_1+2\lambda_2)e^{-(\lambda_1+2\lambda_2)x} + (\lambda_1+\lambda_2)e^{-2(\lambda_1+\lambda_2)x}\big], \end{aligned}$$

respectively. Similarly, we can write the probability density function of J_2 as

$$\begin{aligned} f_{J_2}(x) = 2\big[&\lambda_1 e^{-2\lambda_1 x} + \lambda_2 e^{-2\lambda_2 x} + (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x} \\ &- (2\lambda_1+\lambda_2)e^{-(2\lambda_1+\lambda_2)x} - (\lambda_1+2\lambda_2)e^{-(\lambda_1+2\lambda_2)x} \\ &+ (\lambda_1+\lambda_2)e^{-2(\lambda_1+\lambda_2)x}\big]. \end{aligned}$$

Now it suffices to prove that the function

$$\xi(x) = \frac{f_{J_2}(x)}{f_{J_1}(x)} \propto \frac{A_1(x)}{A_2(x)}$$

is increasing in $x \in \Re_+$, where

$$\begin{aligned} A_1(x) = {} &\lambda_1 e^{(-\lambda_1+\lambda_2)x} + \lambda_2 e^{(\lambda_1-\lambda_2)x} - (2\lambda_1+\lambda_2)e^{-\lambda_1 x} \\ &- (\lambda_1+2\lambda_2)e^{-\lambda_2 x} + (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x} + (\lambda_1+\lambda_2) \end{aligned}$$

and

$$\begin{aligned} A_2(x) = {} &-(2\lambda_1+\lambda_2)e^{-\lambda_1 x} - (\lambda_1+2\lambda_2)e^{-\lambda_2 x} \\ &+ (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x} + 2(\lambda_1+\lambda_2). \end{aligned}$$

Taking derivative of $\xi(x)$ with respect to x, we have

$$\begin{aligned} \xi'(x) \overset{\text{sgn}}{=} {} &2\lambda_1(\lambda_1+\lambda_2)(-\lambda_1+\lambda_2)e^{2\lambda_2 x} + 2\lambda_2(\lambda_1+\lambda_2) \\ &\times(\lambda_1-\lambda_2)e^{2\lambda_1 x} + 3\lambda_1(\lambda_1^2+\lambda_1\lambda_2-\lambda_2^2)e^{\lambda_2 x} \\ &+ 3\lambda_2(\lambda_2^2+\lambda_1\lambda_2-\lambda_1^2)e^{\lambda_1 x} - \lambda_1\lambda_2(2\lambda_1+\lambda_2)e^{(-\lambda_1+2\lambda_2)x} \\ &- \lambda_1\lambda_2(\lambda_1+2\lambda_2)e^{(2\lambda_1-\lambda_2)x} + 2\lambda_1\lambda_2(\lambda_1+\lambda_2)e^{(-\lambda_1+\lambda_2)x} \\ &+ 2\lambda_1\lambda_2(\lambda_1+\lambda_2)e^{(\lambda_1-\lambda_2)x} - (\lambda_1+\lambda_2)^3 \\ = {} &\varphi_1(x), \quad \text{say}. \end{aligned}$$

Notice that $\varphi_1(0) = 0$; hence it suffices if we could show that the derivative of $\varphi_1(x)$ is nonnegative. Observe that

$$\begin{aligned} \varphi_1'(x) \overset{\text{sgn}}{=} {} &4(\lambda_1+\lambda_2)(-\lambda_1+\lambda_2)e^{(3\lambda_2-2\lambda_1)x} \\ &+ 4(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)e^{\lambda_2 x} \\ &+ 3(\lambda_1^2+\lambda_1\lambda_2-\lambda_2^2)e^{2(\lambda_2-\lambda_1)x} \\ &+ 3(-\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)e^{(\lambda_2-\lambda_1)x} \\ &- (2\lambda_1+\lambda_2)(-\lambda_1+2\lambda_2)e^{3(\lambda_2-\lambda_1)x} \\ &- (\lambda_1+2\lambda_2)(2\lambda_1-\lambda_2) - 2(\lambda_1+\lambda_2)(\lambda_1-\lambda_2) \\ &\times e^{(2\lambda_2-3\lambda_1)x} + 2(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)e^{-\lambda_1 x} \\ = {} &\varphi_2(x), \quad \text{say}. \end{aligned}$$

It can be readily verified that $\varphi_2(0) = 0$, and we need to show that $\varphi_2'(x) \ge 0$. Taking derivative of $\varphi_2(x)$ with respect to x, we have

$$\begin{aligned} \varphi_2'(x) \overset{\text{sgn}}{=} {} &4(\lambda_1+\lambda_2)(2\lambda_1-3\lambda_2)e^{\lambda_2 x} + 4\lambda_2(\lambda_1+\lambda_2)e^{(2\lambda_1-\lambda_2)x} \\ &+ 3(\lambda_1^2-\lambda_1\lambda_2-\lambda_2^2)e^{(\lambda_1-\lambda_2)x} \\ &- 3(2\lambda_1+\lambda_2)(\lambda_1-2\lambda_2)e^{(\lambda_2-\lambda_1)x} \\ &- 2(\lambda_1+\lambda_2)(2\lambda_2-3\lambda_1)e^{-\lambda_1 x} \end{aligned}$$

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