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Non-linear equity portfolio variance reduction under a mean–variance framework—A delta–gamma approach

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ABSTRACT

To examine the variance reduction from portfolios with both primary and derivative assets we develop a mean–variance Markovitz portfolio management problem. By invoking the delta–gamma approximation we reduce the problem to a well-posed quadratic programming problem. From a practitioner's perspective, the primary goal is to understand the benefits of adding derivative securities to portfolios of primary assets. Our numerical experiments quantify this variance reduction from sample equity portfolios to mixed portfolios (containing both equities and equity derivatives).

1. Introduction

The main objective in portfolio management is the tradeoff between risk and return. Markovitz, [9,10] studied the problem of maximizing portfolio expected return for a given level of risk, or equivalently minimizing risk for a given expected return. One limitation of Markovitz's model, however, is that it considers only portfolios of primary assets. Mixed portfolios have been a topic of recent research from different perspectives with varying success. We list a few of the known results, and then describe our results in relation to the current research.

Recently, [11,1] looked at the optimal management of portfolios containing primary and derivative assets. In [11], the author introduced a technique for optimizing CVaR (Conditional Value at Risk) of a portfolio. The paper [1] observes that CVaR minimization for a portfolio of derivative securities is ill-posed. Furthermore, [1] has shown that this predicament can be overcome by including transaction costs.

[2,3,6] considered portfolio optimization with non-standard asset classes. In particular, [2] looked at the problem of maximizing expected exponential utility of terminal wealth under a continuous time model by trading a static position in derivative securities and a dynamic position in stocks. Separately, in a one period model, [3] analyzed the optimal investment and equilibrium pricing of primary and derivative instruments. Additionally, [6] has shown how to approximate dynamic positions in options by minimizing the mean-squared error.

To the best of our knowledge, this paper is the first work to consider the mean–variance Markovitz portfolio management problem in a one period model with derivative assets. For a portfolio containing many assets (primary and derivative) the estimation of the correlation matrix can be challenging. Practitioners often solve this difficulty by projecting portfolios onto a reduced set of factors. Projection methods motivate our approach to the mean–variance problem. However, if parametric approaches are used (we work in a multivariate normally distributed returns framework), this projection method creates another problem. Since projections are often non-linear, we must overcome non-linearities by the delta–gamma approximation.

The delta–gamma approximation is well known and often used in risk management and portfolio hedging. In industry practice this approximation works well for sufficiently small time intervals. By performing the delta–gamma approximation, the portfolio management problem with derivative assets is reduced to a quadratic program; however, the covariance matrix of the factors may not be positive definite. Since data are usually built from inconsistent datasets, this issue appears in some financial optimization problems. For example, for portfolios of stocks, the sample correlation matrix is just an approximate correlation, and hence need not be positive definite. This problem is addressed by [5,7]. These works focused on the extraction of a positive semi-definite variance–covariance matrix, obtained through the solution of a second-order conic mathematical programming problem. It is a way to convexify an a priori non convex problem. In [5,7], the smallest distortion of the original matrix which

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satisfies the desired properties (e.g., being a correlation matrix) is obtained by using Frobenius norm.

Our main motivation is to investigate the variance reduction portfolios achieved through the addition of derivative assets compared against straight equity portfolios. Options can be considered as a type of portfolio insurance. We postulate that by allowing investment in this asset class, one should be able to reduce the risk profile of optimal mean–variance portfolios (here risk is measured as variance). Therefore, it appears only important to study the risk reduction due to investing in options. We explore the size of this risk reduction, and how the risk reduction profile varies across different derivative structures.

To address these questions we implement several numerical experiments. One finding is that the largest variance reduction is obtained by adding options on one stock. It is interesting to point out that the optimal portfolio variance reduction is a unimodal function of annual returns (it increases for small values of annual returns, reaches its peak and then decreases for larger values of annual returns). The maximum variance reduction is ∼85%–90%, and it occurs with a portfolio return per annum of ∼12%–15%.

Our results can be applied to the problem of pricing and hedging in incomplete markets. For instance, we can consider instruments written on non-tradable factors (e.g., temperature), and they can be hedged with tradable instruments which are highly correlated (this procedure is called cross hedging). Take weather derivatives (e.g., HDD or CDD) as an example; energy prices are considered as the traded correlated instrument (in California a high correlation can be observed between temperature and energy prices). Perfect hedging is not possible in this paradigm. Minimizing the variance of the hedging error can be captured as a special case of mean–variance optimization problem for a portfolio of primary and derivative instruments. A survey paper on mean–variance hedging and mean–variance portfolio selection is explored in [12].

Another possible application of our results is the hedging of long maturity instruments with short maturity ones. As is well known, the market for long maturity instruments is illiquid, thus the issuers use (static) hedging portfolios of the more liquid short maturity instruments. The interested reader can Ref. [4].

This paper is organized as follows. In Section 2 we present the model; Section 3 introduces the delta–gamma approximation; Section 4 presents the reduction to quadratic programs. Numerical experiments are provided in Section 5; and Section 6 concludes our work.

2. The model

Portfolio returns are derived from the return of individual positions; however, in practice, it is not advisable to model the positions individually due to the latent correlation structure. If we have m instruments in our portfolio, we would need m separate volatilities, plus data on $\frac{m(m-1)}{2}$ correlations, so in total $\frac{m(m+1)}{2}$ pieces of information. For large m this may be difficult.

The resolution is to map m instruments onto a reduced set of risk factors, n. The mapping can be non-linear (e.g., BS (Black–Scholes formula) for options). Let us assume that the factors are represented by a stochastic vector process $S = (S_1, S_2, \dots, S_n)$, which at all times $t \in (0, \infty)$ is assumed to be of the form

$$S(t) = \Sigma W_t \tag{2.1}$$

where Σ is the variance–covariance matrix, which we take to be positive definite (the methodology proposed in [5,7] can be applied when the positive definite assumption fails), and W_t is a standard Brownian motion on a canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The portfolio value at time t, denoted by $V(S, t)$, is of the form

$$V(S, t) = \sum_{k=1}^{m} x_k(t) V_k(S, t), \tag{2.2}$$

where $V_k(S, t)$, $k = 1, \dots, m$, represents the value of the individual instruments (mapped onto the risk factors), and $x_k(t)$, $k = 1, \dots, m$, stands for the number of shares of instrument k held in the portfolio at time t. We choose the portfolio mix $x_k(t)$, $k = 1, \dots, m$, such that the portfolio return, ΔV, over time interval $[t, t + \Delta t]$,

$$\Delta V = V(S + \Delta S, t + \Delta t) - V(S, t), \tag{2.3}$$

is optimized as described below. It turns out to be more convenient to work with the vector of actual proportions of wealth invested in the different assets, thus, at time $t \in (0, \infty)$, we introduce portfolio weights $w_k(t)$, $k = 1, \dots, m$, by

$$w_k(t) = \frac{x_k(t)}{V(S, t)}, \quad k = 1, \dots, m. \tag{2.4}$$

In the following, we posit the Markowitz mean–variance type problem: given some exogenous benchmark return, $r_e(t)$, at time t an investor wants to choose among all portfolios having the same return, $r_e(t)$, the one with minimal variance, $\mathrm{Var}(\Delta V)$:

$$\text{(P1)} \ \min_w \mathrm{Var}(\Delta V)$$
$$\text{s.t. } \mathrm{E}(\Delta V) = r_e(t),$$
$$\sum_{k=1}^{m} w_k(t) V_k(S, t) = 1.$$

Another possible portfolio management problem is to choose the portfolio with minimal variance:

$$\text{(P2)} \ \min_w \mathrm{Var}(\Delta V)$$
$$\text{s.t. } \sum_{k=1}^{m} w_k(t) V_k(S, t) = 1.$$

There are some difficulties in solving (P1) and (P2). First, we may not be able to determine the moments of ΔV since ΔV non-linearly depends on changes in the underlying factors. Moreover, it is not obvious what distribution ΔV would follow—even if we perfectly learnt the pdf of ΔS. If we only required the moments of ΔV, the situation would not improve since the integration of moments might be intractable. One way out of this predicament is to use the delta–gamma approximation.

3. Delta–gamma approximation

The delta–gamma approximation states that a portfolio change during a short time period resulting from the change of underlying factors can be approximated by some second order polynomial function, the coefficients of which are given by the portfolio's sensitivities, such as the portfolio delta, gamma and theta. This approximation is an important tool in risk management and hedging; for instance, to hedge a portfolio of derivatives with respect to the underlying's change, the delta–gamma approximation is employed to match sensitivities of the portfolio with those of the hedging instruments.

Mathematically speaking, this approximation is a second order Taylor expansion of the portfolio's change, ΔV, over the time interval $[t, t + \Delta t]$:

$$\Delta V \approx \delta V = \frac{\partial V}{\partial t} \Delta t + \delta^T \Delta S + \frac{1}{2} \Delta S^T \Gamma \Delta S, \tag{3.1}$$

where

$$\delta_i = \frac{\partial V}{\partial S_i}, \qquad \Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}, \quad i = 1, \dots, n.$$