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Complete description for the spanning tree problem with one linearised quadratic term

Anja Fischer [a,*], Frank Fischer [b]

[a] Technische Universität Dortmund, Fakultät für Mathematik, D-44227 Dortmund, Germany

[b] Technische Universität Chemnitz, Fakultät für Mathematik, D-09107 Chemnitz, Germany

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ABSTRACT

Given an edge-weighted graph the minimum spanning tree problem (MSTP) asks for a spanning tree of minimal weight. The complete description of the associated polytope is well-known. Recently, Buchheim and Klein suggested studying the MSTP with one quadratic term in the objective function resp. the polytope arising after linearisation of that term, in order to better understand the MSTP with a general quadratic objective function. We prove a conjecture by Buchheim and Klein (2013) concerning the complete description of the associated polytope.

1. Introduction and models

Let $G = (V, E)$ be an undirected, simple, complete, edge-weighted graph with node set V, $|V| = n$, set of edges E and weight function $c: E \to \mathbb{R}$. Then the *minimum spanning tree problem* (MSTP) asks for a spanning tree in G with minimal total weight,

$$\begin{aligned} &\text{minimise} \quad c(T) \\ &\text{subject to} \quad T \subseteq G \text{ is a spanning tree,} \end{aligned}$$

where $c(X) := \sum_{e \in E(X)} c(e)$, $X \subseteq G$, with $E(X) := \{e = \{u, v\}: u, v \in X, u \neq v\}$. It is well-known that using a binary variable for each edge $e \in E$ indicating whether the edge is contained in the spanning tree or not a linear integer formulation reads

$$\text{minimise} \quad \sum_{e \in E} c(e) \cdot x(e)$$

$$\text{subject to} \quad -x(E) = 1 - |V|, \tag{1}$$

$$-x(E(S)) \geq 1 - |S|, \quad \emptyset \neq S \subsetneq V, \tag{2}$$

$$x(e) \in \{0, 1\}, \quad e \in E. \tag{3}$$

Edmonds [5] proved that replacing $x(e) \in \{0, 1\}$, $e \in E$, by $x(e) \geq 0$ yields a complete description of the associated polytope. So we get a linear optimisation problem formulation (LP) for the MSTP. Its corresponding dual linear program (DP) reads

$$\text{maximise} \quad \sum_{\emptyset \neq S \subseteq V} (1 - |S|) z_S$$

$$\text{subject to} \quad -\sum_{S: e \subseteq S \subseteq V} z_S \leq c(e), \quad e \in E, \tag{4}$$

$$z_V \text{ free}, \quad z_S \geq 0, \emptyset \neq S \subsetneq V.$$

Although the linear spanning tree problem and its associated polytope are well understood, not much is known if the objective function depends on products of edge-variables, *i.e.*, if we want to optimise

$$\sum_{e \in E} c(e) \cdot x(e) + \sum_{e, f \in E, e \neq f} c_q(e, f) \cdot x(e) \cdot x(f)$$

with additional weight function $c_q: E \times E \to \mathbb{R}$. The so called *Quadratic Minimum Spanning Tree Problem* (QMSTP) is known to be $\mathcal{NP}$-hard [1]. This is analogous to the *Assignment Problem*, which can be solved efficiently and whose polyhedral structure is well-known, and the *Quadratic Assignment Problem* (see, *e.g.*, [8]), which is one of the computationally most challenging combinatorial optimisation problems. Some branch-and-bound algorithms and heuristics for the QMSTP were presented, *e.g.*, in [1,9,4]. However, not much is known about the structure of the polytope that arises after a linearisation of $x(e)x(f)$, $e, f \in E$, $e \neq f$, by introducing new variables $y(e, f)$, $e, f \in E$, $e \neq f$. In order to better

* Corresponding author.
E-mail addresses: anja.fischer@mathematik.tu-dortmund.de (A. Fischer), frank.fischer@mathematik.tu-chemnitz.de (F. Fischer).

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understand the polyhedral structure of the QMSTP and of combinatorial optimisation problems with a quadratic objective function in general, Buchheim and Klein [3,2] suggested considering the special case of the QMSTP resp. of a combinatorial optimisation problem with exactly one quadratic term in the objective function. Because the MSTP is polynomially-solvable QMSTP-1 (QMSTP with one quadratic term) can be solved in polynomial time, too, and by the well-known "optimisation equals separation" result [6], we can hope to fully characterise the polytope of the linearised version of QMSTP-1. Furthermore, the separation algorithms for valid inequalities or facets of QMSTP-1 may also be useful for solving the general QMSTP because valid inequalities of the one-monomial-case remain valid for the general case. First computational experiments in [3,2] also indicate this behaviour.

QMSTP-1 can be formally described as follows. Let $u_1, v_1, u_2, v_2 \in V$ with $u_1v_1, u_2v_2 \in E$, $u_1v_1 \neq u_2v_2$, either $\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset$ or $v_1 = v_2$, $u_1 \neq u_2$, and $\bar{c} \in \mathbb{R}$ be the *monomial weight*. Then QMSTP-1 reads

$$\text{minimise} \quad q(T) := c(T) + \begin{cases} \bar{c}, & u_1v_1, u_2v_2 \in T, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{subject to} \quad T \subseteq G \text{ is a spanning tree.}$$

In [3] the case $v_1 = v_2$ is called the *connected case* because the two edges u_1v_1 and u_2v_2 share a common node, otherwise it is called the *unconnected case*. In most parts we will not distinguish between the two cases.

In this article we will prove that the following equations and inequalities are a complete description of the integer polytope if we linearise the monomial $x(u_1v_1) \cdot x(u_2v_2)$ by introducing a new variable y. Let

$$\mathcal{S} := \{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1v_1, u_2v_2 \in E(S, S')\}$$

with $E(X, Y) := \{e = \{u, v\} \in E : u \in X, v \in Y\}$. Then (QP) reads

$$\text{minimise} \quad \sum_{e \in E} c(e) \cdot x(e) + \bar{c} \cdot y$$

subject to (1), (2), $x \geq 0$

$$-x(E(S) \cup E(S')) - y \geq 2 - |S| - |S'|, \quad (S, S') \in \mathcal{S}, \tag{5}$$

$$x_{u_iv_i} - y \geq 0, \quad i \in \{1, 2\}, \tag{6}$$

$$y - x_{u_1v_1} - x_{u_2v_2} \geq -1, \tag{7}$$

$$y \geq 0. \tag{8}$$

Let $\mathcal{F}$ be a family of sets, then we write $z(\mathcal{F}) = \sum_{F \in \mathcal{F}} z_F$ and $\bar{z}(\mathcal{F}) = \sum_{F \in \mathcal{F}} \bar{z}_F$, respectively. So the dual problem (DQP) is

$$\text{maximise} \quad \sum_{\emptyset \neq S \subseteq V} (1 - |S|) z_S + \sum_{(S,S') \in \mathcal{S}} (2 - |S| - |S'|) \bar{z}_{(S,S')} - \zeta_y \tag{9}$$

$$\text{subject to} \quad - \sum_{S : e \subseteq S \subseteq V} z_S - \sum_{\substack{(S,S') \in \mathcal{S}: \\ e \in E(S) \cup E(S')}} \bar{z}_{(S,S')} \leq c(e), \quad e \in E \setminus \{u_1v_1, u_2v_2\}, \tag{10}$$

$$- \sum_{S : u_iv_i \subseteq S \subseteq V} z_S + \zeta_{u_iv_i} - \zeta_y \leq c(u_iv_i), \quad i \in \{1, 2\}, \tag{11}$$

$$-\bar{z}(\mathcal{S}) - \zeta_{u_1v_1} - \zeta_{u_2v_2} + \zeta_y \leq \bar{c}, \tag{12}$$

$$z_S \geq 0, \quad \emptyset \neq S \subsetneq V, \quad \bar{z}_{(S,S')} \geq 0, \quad (S, S') \in \mathcal{S}, \quad z_V \text{ free}, \tag{13}$$

$$\zeta_{u_1v_1}, \zeta_{u_2v_2}, \zeta_y \geq 0. \tag{14}$$

Indeed, Buchheim and Klein conjectured that in the unconnected case the model (QP) above provides a complete description of QMSTP-1. In the connected case, their conjecture looks a bit different. It says that apart from the standard linearisation (6)–(8) and the formulation of the MSTP (1)–(2), $x \geq 0$ one only needs

$$-x(E(S)) - y \geq 1 - |S|, \quad S \subset V, u_1, u_2 \in S, v_1 = v_2 \notin S,$$

for a complete description. If we can show that (QP) is a complete description this conjecture follows because then $\{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1, v_2 \in S, u_2, v_1 \in S'\} = \emptyset$ and inequalities (5) with $(S, S') \in \{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1, u_2 \in S, v_1 = v_2 \in S', |S'| > 1\}$ are implied by (5) with $(S, S') \in \{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1, u_2 \in S, S' = \{v_1\}\}$ and (2). Note that in the meantime Buchheim and Klein independently proved the above-mentioned conjectures. A complete proof for the connected case can be found in [2].

2. Notation and previous results

In the following we write $[k]$ instead of $\{1, \ldots, k\}$, $k \in \mathbb{N}$. We denote the objective value of a spanning tree T w. r. t. $\tilde{c} : E \to \mathbb{R}$ by

$$v_{\text{LP}}(\tilde{c}, T) = \sum_{e \in E(T)} \tilde{c}(e) \quad \text{and} \quad v_{\text{DP}}(z) := \sum_{S : \emptyset \neq S \subseteq V} (1 - |S|) z_S$$

denotes the value of a solution z of (DP) with $z = (z_S)_{S:\emptyset \neq S \subseteq V}$. The following result follows from [5] and can, *e.g.*, be found in [7] (proof of Theorem 6.13).

Lemma 1 ([5,7]). *Let T be a minimum spanning tree (MST) in G w. r. t. $\tilde{c} : E \to \mathbb{R}$ computed by the greedy algorithm. Let $f_1, \ldots, f_{|V|-1}$ be the edges selected by the (best-in) greedy algorithm in order and denote by $X_k \subseteq V$, $k \in [|V| - 1]$, the nodes of the connected component of $(V, \{f_1, \ldots, f_k\})$ that contains f_k. Furthermore, let $s(k) \in [|V| - 1]$, $k \in [|V| - 2]$, denote the smallest index greater than k so that $f_{s(k)} \cap X_k \neq \emptyset$. Then the dual solution*

$$z^*(\tilde{c}, T) = (z_S)_{S:\emptyset \neq S \subseteq V},$$

$$\textit{with} \quad z_S := \begin{cases} \tilde{c}(f_{s(k)}) - \tilde{c}(f_k), & S = X_k, k < |V| - 1, \\ -\tilde{c}(f_{|V|-1}), & S = X_{|V|-1} = V, \\ 0, & \textit{otherwise}, \end{cases}$$

is an optimal solution of (DP). In particular, for any edge $e \in E$ there holds $\text{lhs}(z, e) := -\sum_{S:e \subseteq S \subseteq V} z_S = \tilde{c}(f_i)$, where $i \in [|V| - 1]$ is the smallest index so that $e \subseteq X_i$.

Remark 2. Note that we may assume, w.l.o.g., that each variable $z_{\{u\}}$, $u \in V$, of the solution $z^*(\tilde{c}, T)$ has an arbitrarily large value, because these variables do not contribute to the objective value and do not appear in any constraint except for $z_{\{u\}} \geq 0$. We will make use of this property later in Corollary 5.

We denote the value of spanning tree T w. r. t. $\tilde{c} : E \to \mathbb{R}$ and weight $\bar{c}$ by

$$v_{\text{QP}}(\tilde{c}, T) = \sum_{e \in E(T)} \tilde{c}(e) + \begin{cases} \bar{c}, & u_1v_1, u_2v_2 \in T, \\ 0, & \text{otherwise}, \end{cases}$$

and the value of a solution $(z, \bar{z}, \zeta)$ of (DQP) with $z = (z_S)_{S:\emptyset \neq S \subseteq V}$, $\bar{z} = (\bar{z}_{(S,S')})_{(S,S') \in \mathcal{S}}$ and $\zeta = (\zeta_y, \zeta_{u_1v_1}, \zeta_{u_2v_2})$ by

$$v_{\text{DQP}}(z, \bar{z}, \zeta) = \sum_{S:\emptyset \neq S \subseteq V} (1 - |S|) z_S + \sum_{(S,S') \in \mathcal{S}} (2 - |S| - |S'|) \bar{z}_{(S,S')} - \zeta_y.$$

3. Complete description

In this section we will prove that (QP) is indeed a complete description of the integer polytope for QMSTP-1. We start with a