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Feedback saddle point solution of counterterror measures and economic growth game

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ABSTRACT

In this paper, the feedback Nash equilibrium solutions of the differential game between counterterror measures and economic growth are investigated. The Hamilton–Jacobi–Isaacs (HJI) equation is used to obtain the feedback saddle point of this zero-sum game. Moreover, the characteristics of the feedback strategies for the government and terrorist organization as well as their relationships with both resource states are analyzed.

1. Introduction

Terrorism is increasingly rampant in recent years. Despite the enhanced counterterror force in all countries, terrorist attacks still occur widely. Therefore, we should think how to effectively suppress terrorism. Is it feasible to merely increase defense expenditure? This problem has been studied by various researchers from their own fields [1–11]. People are endeavoring to solve this hot issue through social investigations, statistical analysis or mathematical proof. Because of the strong hostility between the government and terrorist organizations, the optimal control theory and game theory are often applied to build relevant models [3,5–11]. Meanwhile, it cannot be neglected that terrorism relates closely to the governmental economy [1,2,4]. On the one hand, terrorism causes certain social turbulences which are mostly targeted at social economy. On the other hand, social progress needs support from economic development, and it is unrealistic for the government to spare all energy in counterterror.

In Ref. [11], the strategic equilibrium considering both counterterror measures and economic development was first put forward. In this differential game model, the open-loop Nash equilibrium solution was discussed, which only relates to time and the initial state. Obviously, the open-loop solution is a relatively simple concept in differential games. Faced with complicated situations such as counterterror, both the government and terrorist organizations shall adjust strategies timely according to the changes in their states. Specifically, the strategies shall relate to the current states. The objective of this paper is to study this more complex solution concept, called the feedback Nash equilibrium.

2. Model and definition

Some symbols and definitions in Ref. [11] are adopted. There are two state functions: $x(t)$ which describes the resources of terrorist organization (RTO) and $y(t)$ which describes the government's resources (economic income). Government and terrorist organization are two players in this differential game. The control function of player 1 (government) $u(t)$, $u(t) \geqslant 0$, is the proportion devoted to counterterror, and $1-u(t)$ is the part devoted to economic development. The detailed analysis of function $u(t)$ is provided in Ref. [11]. $v(t) \geqslant 0$ is the control function of player 2 (terrorist organization), which measures the intensity of attacks. The dynamic system can be written as

$$\begin{cases} \dot{x}(t) = bx(t) - (u(t)y(t))^a (v(t))^s, & x(t_0) = x_0 > 0, \\ \dot{y}(t) = \alpha(1-u(t))y(t) - f v(t), & y(t_0) = y_0 > 0 \end{cases} \tag{1}$$

where $0 < a < 1 < s, b > 0, \alpha > 0$ and $f > 0$. x_0 and y_0 denote the initial states of $x(t)$ and $y(t)$ at $t = t_0$. We assume $x(t) \geqslant 0,\ y(t) > 0$ on $[0, \infty)$.

We consider a zero-sum differential game. Let the objective function of the two players be

$$J(u, v) = \int_{t_0}^{\infty} \mathrm{e}^{-\rho(t-t_0)}[ly(t) - cx(t) - kv(t)]\mathrm{d}t, \tag{2}$$

where l, c, k, ρ are positive constants. Player 1 expects to control $u(t)$ so as to maximize the objective function $J(u, v)$, while player 2 expects to control $v(t)$ so as to minimize $J(u, v)$. ρ is the discount rate. Let $\rho > b$, $\rho > \alpha$. In fact, the government hopes to get more

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income $y(t)$ and less RTO stock $x(t)$, and to lower the intensity of terrorist attacks by counterterror measures. The object of terrorist organizations is just the reverse. The assumptions of the model were elaborated in Ref. [11].

We give the definition of the feedback Nash equilibrium of infinite horizon differential games [12,13]. Now consider the game

$$\max_{u_i} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} g^i[x(t), u_1(t), u_2(t), \ldots, u_n(t)]\mathrm{d}t,$$
$$\text{for } i \in N, \tag{3}$$

subject to the dynamics

$$\dot{x}(t) = h[x(t), u_1(t), u_2(t), \ldots, u_n(t)], \qquad x(t_0) = x_0. \tag{4}$$

Definition 1. For the differential game (3)–(4), an n-tuple of strategies

$$\{u_i^*(t) = \phi_i^*(x) \in U_i, \text{ for } i \in N\}$$

constitutes a feedback Nash equilibrium solution if there exist functionals $V^i(\theta, x)$ defined on $[0, +\infty) \times R^n$ and satisfying the following relations for each $i \in N$:

$$V^i(\theta, x) = \int_{\theta}^{+\infty} e^{-\rho(t-t_0)} g^i[x^*(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_n^*(\eta_t)]\mathrm{d}t$$
$$\geqslant \int_{\theta}^{+\infty} e^{-\rho(t-t_0)} g^i[x^{[i]}(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_{i-1}^*(\eta_t), \phi_i(\eta_t),$$
$$\phi_{i+1}^*(\eta_t), \ldots, \phi_n^*(\eta_t)]\mathrm{d}t, \quad \forall \phi_i \in \Gamma^i, \; x \in R^n,$$

where on the interval $[0, +\infty)$,

$$\dot{x}^{[i]}(t) = h[x^{[i]}(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_{i-1}^*(\eta_t)\phi_i(\eta_t), \phi_{i+1}^*(\eta_t),$$
$$\ldots, \phi_n^*(\eta_t)], \qquad x^{[i]}(\theta) = x;$$
$$\dot{x}^*(t) = h[x^*(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_n^*(\eta_t)], \qquad x^*(\theta) = x,$$

and η_t stands for the data set $\{x(t), x_0\}$.

Based on this definition, if a feedback Nash equilibrium solution of a differential game with duration $[0, +\infty)$, its restriction to the time interval $[\theta, +\infty)$ provides a feedback Nash equilibrium solution to the same differential game defined on the shorter time interval $[\theta, +\infty)$ with the initial state taken as $x(\theta)$, and this being so for all $\theta \geqslant 0$. Hence, a feedback Nash equilibrium solution is strongly time consistent. In other words, the feedback Nash equilibrium solution only depends on the current state rather than the previous state or the initial state x_0. In this definition, ϕ_i^* is a feedback strategy related to state, and that they induce controls $u(t)$.

3. Feedback saddle point

As model (1)–(2) is an infinite horizon zero-sum differential game, before solving the HJI equation of this game, the problem can be simplified for processing. If

$$V(\theta, x, y) = e^{-\rho(\theta-t_0)} W(x, y),$$
$$\text{for } x(\theta) = x = x_\theta^* = x^*(\theta), \quad y(\theta) = y = y_\theta^* = y^*(\theta),$$

where $W(x, y)$ only depends on the current state x, y,

$$W(x, y) = \int_{\theta}^{+\infty} e^{-\rho(t-\theta)}[ly^*(t) - cx^*(t) - k\phi_2^*(\eta_t)]\mathrm{d}t$$

then

$$\frac{\partial V(\theta, x, y)}{\partial \theta} = -\rho e^{-\rho(\theta-t_0)} W(x, y),$$
$$\frac{\partial V(\theta, x, y)}{\partial x} = e^{-\rho(\theta-t_0)} \frac{\partial W(x, y)}{\partial x},$$
$$\frac{\partial V(\theta, x, y)}{\partial y} = e^{-\rho(\theta-t_0)} \frac{\partial W(x, y)}{\partial y}.$$

Now we discuss the saddle points of the differential game. We introduce the following proposition.

Proposition 2. *The feedback saddle point* $(u^*(t), v^*(t)) = (\phi_1^*(x, y), \phi_2^*(x, y))$ *of the differential game* (1)–(2) *is given by*

$$\phi_1^*(x_\theta^*, y_\theta^*) = \frac{a(k + fB)}{\alpha s B y_\theta^*} \left(-\frac{\alpha^a s^{a-1} B^a}{a^a (k + fB)^{a-1} A}\right)^{\frac{1}{a+s-1}}, \tag{5}$$

$$\phi_2^*(x_\theta^*, y_\theta^*) = \left(-\frac{\alpha^a s^{a-1} B^a}{a^a (k + fB)^{a-1} A}\right)^{\frac{1}{a+s-1}}. \tag{6}$$

And

$$W(x, y) = Ax + By + C, \tag{7}$$

where

$$A = -\frac{c}{\rho - b},$$
$$B = \frac{l}{\rho - \alpha},$$
$$C = -\frac{(a + s - 1)(k + fB)}{s\rho} \left(-\frac{\alpha^a s^{a-1} B^a}{a^a (k + fB)^{a-1} A}\right)^{\frac{1}{a+s-1}}.$$

From Proposition 2, the feedback saddle point of the game that constantly adjusts with the state variation can be obtained. It can be learnt from (5)–(6) that the feedback strategy ϕ_1^* of player 1 or the government merely relates to the state variable y_θ^*, indicating that only the government's economic state affects the proportion of counterterror investment. Moreover, the proportion of counterterror investment decreases when economic income increases; therefore, the two are in an inverse relationship. However, it can be learnt from the relational expression that $\phi_1^* y_\theta^*$ is constant, meaning however the economic state changes, the optimal counterterror investment shall be kept the same. In other words, counterterror measure is a normal input which shall not affect the economic state. Furthermore, formula (5) indicates ϕ_1^* does not relate to the state x(RTO stock). That is, the government does not fully know the resource state of the terrorist organization, which accords with the information asymmetry between two parties.

The feedback strategy ϕ_2^* of player 2 or the terrorist organization is a constant, which has nothing to do with states x or y. The RTO amount and the current economic state of the government will not affect the intensity of terrorist attacks. Even in the adverse condition for RTO stock, the optimal strategy of terrorist organizations is still to constantly attack the government. This exactly reflects the fearlessness and unscrupulousness of terrorism.

In proof of Proposition 2, the relational expression of (12) can be obtained, indicating that the counterterror investment is in proportion to the attack intensity of terrorist organizations. Stronger terrorist attacks will result in a larger counterterror investment, which in turn will lead to stronger terrorist attacks. Since the counterterror investment does not relate to RTO (state x), the government's counterterror measures are more sensitive to terrorist attacks than to RTO. In fact, the government's counterterror investment usually will be immediately improved to a higher level only when it suffers from a great attack. However, the government ignores weakening the RTO stock in normal time.

Here we prove Proposition 2.

Proof. According to the above discussion, the HJI equation can be written as follows