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The discrete moment problem with fractional moments

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ABSTRACT

Discrete moment problems (DMP) with integer moments were first introduced by Prékopa to provide sharp lower and upper bounds for functions of discrete random variables. Prékopa also developed fast and stable dual type linear programming methods for the numerical solutions of the problem. In this paper, we assume that some fractional moments are also available and propose basic theory and a solution method for the bounding problems. Numerical experiments show significant improvement in the tightness of the bounds.

1. Introduction

Let X be a discrete random variable, the possible values of which are known to be the numbers $z_0 < z_1 < \cdots < z_n$ and

$$p_i = P(X = z_i), \quad i = 0, 1, \ldots, n. \tag{1.1}$$

Given the knowledge of some power moments $\mu_k = E(X^k)$, $k = 1, \ldots, m$, where $m < n$, the discrete moment bounding problem provides us with the sharp lower and upper bounds on a linear functional, defined on the unknown probability distribution $\{p_i\}$. It can be formulated as the following LP:

$$\min(\max) \sum_{i=0}^{n} f_i p_i$$

subject to

$$Ap = b \tag{1.2}$$

$$p \geq 0,$$

where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_n \\ & & \cdots & \\ z_0^m & z_1^m & \cdots & z_n^m \end{pmatrix}, \qquad b = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \cdots \\ \mu_m \end{pmatrix}, \tag{1.3}$$

and $f_i = f(z_i),\ i = 0, \ldots, n$.

Discrete moment problems (DMPs) were introduced and studied by Prékopa (see, e.g. [10–13]). In those papers, the author used linear programming techniques to develop theory and numerical solution of the optimization problems. The optimization methods are of dual type and are in close relationship with the dual method of Lemke [4] for the solution of the general linear programming problem. They are stable and fast thanks to the discovery of the structures of dual feasible bases. DMPs have been used extensively in data mining, biology, finance, engineering, etc. For further information on DMPs and their generalizations, we refer to Prékopa [14–18], Subasi et al. [20], Mádi-Nagy and Prékopa [7] and Mádi-Nagy [5,6]. Fractional moments have been used within the context of the (discrete) maximum entropy problems to find probabilities in a distribution, where some of the fractional moments are given (see, e.g. Novi Inverardi and Tagliani [8]). In this paper, we present and solve the bounding problems in the spirit of the discrete moment problem proposed by Prékopa. The discrete fractional moment problem can be formulated as:

$$\min(\max) \sum_{i=0}^{n} f_i(z_i) p_i$$

subject to

$$\sum_{i=0}^{n} z_i^{\alpha_k} p_i = \mu_k \quad k = 0, \ldots, m \tag{1.4}$$

$$p_i \geq 0 \quad i = 0, \ldots, n,$$

where $\alpha_k,\ k = 0, 1, 2, \ldots$ are positive numbers. We consider three objective functions:

(1) The function $f(z)$ is absolutely monotonic. The optimum values of problems (1.4) give sharp lower and upper bounds for $E(f(X))$.

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(2) $f_r = 1, f_i = 0$, if $i \neq r$, for some $0 \leq r \leq n$. The optimum values of problems (1.4) give sharp lower and upper bounds for $P(X = z_i)$.
(3) $f_0 = \cdots = f_{r-1} = 0, f_r = \cdots = f_n = 1$, for some $1 \leq r \leq n$. The optimum values of problems (1.4) give sharp lower and upper bounds for $P(X \geq z_r)$.

The paper is organized as follows. Section 2 presents some basic notions and theorems for the discrete moment problems with fractional moments. In Section 3 basis structure theorems are presented for the above mentioned objective functions. In Section 4 we provide a detailed description of the dual method that solves the problem and a procedure to estimate fractional moments. Numerical results are reported in Section 5.

2. Basic notions and theorems

A function $f(z)$ is said to be absolutely monotonic on $(0, \infty)$ if it has derivatives of all orders and

$$f^{(k)}(z) \geq 0, \quad z \in (0, \infty),\ k = 0, 1, 2, \ldots. \tag{2.1}$$

Theorem 2.1. *Assume that $f(z)$ is an absolutely monotonic function on $(0, \infty)$, $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_m \leq 1$ and $0 < z_0 < z_1 < \cdots < z_m$. Then the following inequality holds:*

$$D(z, f) = \begin{vmatrix} z_0^{\alpha_0} & z_1^{\alpha_0} & \cdots & z_m^{\alpha_0} \\ & & \vdots & \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} \\ z_0^{\alpha_m} & z_1^{\alpha_m} & \cdots & z_m^{\alpha_m} \\ f(z_0) & f(z_1) & \cdots & f(z_m) \end{vmatrix} > 0. \tag{2.2}$$

Proof. It is well-known (Bernstein [1]) that any absolutely monotonic function can be expressed as a series of polynomials with nonnegative coefficient. Thus, we can rewrite $f(z)$ as follows:

$$f(z) = \sum_{i=0}^{\infty} c_i z^i. \tag{2.3}$$

Then the original determinant $D(z, f)$ can be represented as:

$$D(z, f) = \sum_{i=0}^{\infty} D(z, c_i z^i). \tag{2.4}$$

The ith term $(i \geq 1)$ in the series (2.4) is c_i times the determinant:

$$\begin{vmatrix} z_0^{\alpha_0} & z_1^{\alpha_0} & \cdots & z_m^{\alpha_0} \\ & & \vdots & \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} \\ z_0^{\alpha_m} & z_1^{\alpha_m} & \cdots & z_m^{\alpha_m} \\ z_0^{i} & z_1^{i} & \cdots & z_m^{i} \end{vmatrix}. \tag{2.5}$$

This is a generalized Vandermonde determinant which is known to be positive (Karlin and Studden [3, p. 9]) and the assertion follows. □

Theorem 2.2. *Assume that $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $0 < z_0 < z_1 < \cdots < z_m$, then we have the following inequality*

$$(-1)^t \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ z_0^{\alpha_0} & z_1^{\alpha_0} & \cdots & z_t^{\alpha_0} & z_{t+1}^{\alpha_0} & \cdots & z_m^{\alpha_0} \\ & & & \vdots & & & \\ z_0^{\alpha_{m-2}} & z_1^{\alpha_{m-2}} & \cdots & z_t^{\alpha_{m-2}} & z_{t+1}^{\alpha_{m-2}} & \cdots & z_m^{\alpha_{m-2}} \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} & \cdots & z_t^{\alpha_{m-1}} & z_{t+1}^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} \end{vmatrix} > 0. \tag{2.6}$$

Proof. It can be easily verified that the above determinant can be simplified as follows:

$$\begin{vmatrix} z_0^{\alpha_0} & z_1^{\alpha_0} - z_0^{\alpha_0} & \cdots & z_{t+1}^{\alpha_0} - z_t^{\alpha_0} & \cdots & z_m^{\alpha_0} - z_{m-1}^{\alpha_0} \\ & & & \vdots & & \\ z_0^{\alpha_{m-2}} & z_1^{\alpha_{m-2}} - z_0^{\alpha_{m-2}} & \cdots & z_{t+1}^{\alpha_{m-2}} - z_t^{\alpha_{m-2}} & \cdots & z_m^{\alpha_{m-2}} - z_{m-2}^{\alpha_{m-2}} \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} - z_1^{\alpha_{m-1}} & \cdots & z_{t+1}^{\alpha_{m-1}} - z_t^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} - z_{m-1}^{\alpha_{m-1}} \end{vmatrix}.$$

We rewrite it as shown in Box I, which can be further reduce to:

$$\prod_{j=0}^{m-1} \alpha_j \int_0^{z_0} \int_{z_0}^{z_1} \cdots \int_{z_{m-2}}^{z_{m-1}} \bar{D} dx_1 dx_2 \ldots dx_{m+1}, \tag{2.7}$$

where

$$\bar{D} = \begin{vmatrix} x_1^{\alpha_0 - 1} & x_2^{\alpha_0 - 1} & \cdots & x_t^{\alpha_0 - 1} & \cdots & x_{m+1}^{\alpha_0 - 1} \\ & & \vdots & & & \\ x_1^{\alpha_{m-2} - 1} & x_2^{\alpha_{m-2} - 1} & \cdots & x_t^{\alpha_{m-2} - 1} & \cdots & x_{m+1}^{\alpha_{m-2} - 1} \\ x_1^{\alpha_{m-1} - 1} & x_2^{\alpha_{m-1} - 1} & \cdots & x_t^{\alpha_{m-1} - 1} & \cdots & x_{m+1}^{\alpha_{m-1} - 1} \end{vmatrix}.$$

It is well known (Karlin and Studden [3, p. 9]) that $\bar{D}$ is positive since it is the determinant of a generalized Vandermonde matrix and the assertion of the theorem follows. □

3. DMP with fractional moments

A generalization of the discrete moment problem, called the totally positive linear programming problem, was introduced in Prékopa [13]. It is the LP:

$$\begin{aligned} \min(\max) \quad & f_0 p_0 + f_1 p_1 + \cdots + f_n p_n \\ \text{s.t.} \quad & a_{00}\, p_0 + \cdots + a_{0n}\, p_n = 1 \\ & a_{10}\, p_0 + \cdots + a_{1n}\, p_n = b_1 \\ & a_{20}\, p_0 + + a_{2n}\, p_n = b_2 \\ & \vdots \\ & a_{m0}\, p_0 + \cdots a_{mn}\, p_n = b_m, \\ & p_i \geqslant 0, \quad i = 1, 2, \ldots, n, \end{aligned}$$

where all $(m+1) \times (m+1)$ submatrices of A and all $(m+2) \times (m+2)$ submatrices of $\binom{f^T}{A}$ have positive determinants, where $A = (a_{ik})$. By the theorems established in the previous section the discrete fractional moment problem belongs to this class and we can apply the dual feasible basis structure theorem in Prékopa [13].

$$\prod_{j=0}^{m-1} \alpha_j \begin{vmatrix} \int_0^{z_0} x_1^{\alpha_0 - 1} dx_1 & \int_{z_0}^{z_1} x_2^{\alpha_0 - 1} dx_2 & \cdots & \int_{z_t}^{z_{t+1}} x_t^{\alpha_0 - 1} dx_t & \cdots & \int_{z_{m-1}}^{z_m} x_{m+1}^{\alpha_0 - 1} \\ & & & \vdots & & \\ \int_0^{z_0} x_1^{\alpha_{m-2} - 1} dx_1 & \int_{z_0}^{z_1} x_2^{\alpha_{m-2} - 1} dx_2 & \cdots & \int_{z_t}^{z_{t+1}} x_t^{\alpha_{m-2} - 1} dx_t & \cdots & \int_{z_{m-1}}^{z_m} x_{m+1}^{\alpha_{m-2} - 1} \\ \int_0^{z_0} x_1^{\alpha_{m-1} - 1} dx_1 & \int_{z_0}^{z_1} x_2^{\alpha_{m-1} - 1} dx_2 & \cdots & \int_{z_t}^{z_{t+1}} x_t^{\alpha_{m-1} - 1} dx_t & \cdots & \int_{z_{m-1}-1}^{z_m} x_{m+1}^{\alpha_{m-1} - 1} \end{vmatrix},$$

Box I.