Operations Research Letters 41 (2013) 726–729

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Determining the fill rate for a periodic review inventory policy with capacitated replenishments, lost sales and zero lead time

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ARTICLE INFO

Article history:
Received 10 June 2013
Received in revised form
16 October 2013
Accepted 22 October 2013
Available online 26 October 2013

Keywords:
Inventory
Periodic review
Fill rate
Capacitated replenishment
Lost sales

ABSTRACT

In this paper we consider a periodic review order-up-to inventory system with capacitated replenishments, lost sales and zero lead time. We consider discrete demand. It is shown that the initial stock levels of the different review periods form a Markov chain and we determine the transition matrix. Furthermore we study for what probability mass functions of the review period demand the Markov chain has a unique stationary distribution. Finally, we present a method to determine the fill rate.

1. Introduction

In several publications the fill rate is discussed for periodic review inventory systems with uncapacitated replenishment. For example Johnson et al. [5] study different fill rate expressions for inventory systems with backorders and normally distributed demand and compare these expressions experimentally via simulation experiments. The fill rate of an uncapacitated periodic review inventory system with backorders and continuous period demand is also studied in [8,10,7]. Sobel [8] discusses besides single-stage systems also multistage systems and similar as in [5] the lead time is assumed to be a multiple of the review period. This is not assumed in Zhang et al. [10] and Silver et al. [7]. In [8,10] general continuous demand and normal demand are considered, and [5,7] focus on normal demand. Guijarro et al. [4] discuss fill rate definitions and expressions for uncapacitated periodic review inventory systems with lost sales and discrete demand. In this paper, however, periodic review inventory systems with a limited replenishment capacity are studied. Unlike [5,8,10,7,4], in this paper the lead time is assumed to be negligible. In a part of [8], capacity is also considered, but in the context of multistage systems with process limitations. In [2,1] finite horizon fill rates are considered and compared with the infinite horizon fill rate.

We consider a single-item inventory system that applies a periodic review order-up-to inventory policy with lost sales and zero lead time. Because of the lost sale assumption and the zero lead time assumption, the inventory position (the number of products on hand minus the number of products backlogged plus the number of products on order) equals the stock level (the number of products on hand). In such an inventory policy the stock level is reviewed periodically and in every review an order is placed to raise the stock level to a fixed level, the order-up-to level s (a positive integer). We assume the demand during one review period (period between two reviews) to be discrete with a given probability mass function. We consider a review period to begin when the order is placed and to end just before the next order is placed. Following characteristics are assumed for the inventory system under study: (i) the order is placed immediately after review; (ii) the lead time is zero, i.e., the order arrives immediately after the order is placed; (iii) the demands during different review periods are independently and identically distributed; (iv) the demand during a particular review period is independent of every stock level at the beginning of a review period that precedes that review period or coincides with that review period; (v) unsatisfied demands result in lost sales; and (vi) replenishment is capacitated with capacity c (a positive integer), i.e., if more than c products are ordered, only c are delivered.

In this paper we determine the fill rate of a periodic review inventory system with capacitated replenishments. A similar problem was already studied by Mapes [6], who determined the service level of a capacitated periodic review inventory system

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http://dx.doi.org/10.1016/j.orl.2013.10.006

approximately by simulation. In this paper a new method to determine the fill rate is presented which is exact given the used fill rate definition and the above stated six assumptions. Similarly as in [9], we define the fill rate of a periodic review inventory system as the proportion of the expected satisfied demand to the expected demand (see (10) for the exact formula). Another definition used in the literature for the fill rate (e.g. in [8,10]) is the expectation of the proportion of the satisfied demand to the demand. According to [2,10], both definitions agree if an infinite horizon is considered.

2. Determination of the fill rate

In this section we will determine the fill rate β of a periodic review order-up-to inventory system with order-up-to level s and replenishment capacity c. We assume $c < s$ because when c is greater than or equal to s replenishment is not capacitated. Let D_t be the random variable associated with the demand during the review period t, I_t the random variable associated with the stock level at the beginning of review period t, f_D the probability mass function of D_t (with the set of the integers as domain and value zero for negative integers) and f_{It} the probability mass function of I_t, for all $t \in \{1, 2, \ldots\}$. We assume the stock level at the beginning of the first review period to be $c, c+1, \ldots$ or s. Because of the used inventory policy, the following holds:

$$I_t = \min\{s, \max\{I_{t-1} - D_{t-1}, 0\} + c\}, \quad \text{for all } t \in \{2, 3, \ldots\}. \tag{1}$$

We continue by first proving four theorems and then presenting a method to find the fill rate based on these theorems. For (finite state) the Markov chain theory we refer to [3], Chapter 4.

Theorem 1. *$I_1, I_2, I_3, \ldots$ is a Markov chain.*

Proof. For proving Theorem 1, we need to prove the following:

$$\begin{aligned} &P(I_t = i_t \,|I_{t-1} = i_{t-1} \cap I_{t-2} = i_{t-2} \cap \cdots \cap I_1 = i_1) \\ &\quad = P(I_t = i_t \,|I_{t-1} = i_{t-1}), \quad \text{for all } t \in \{2, 3, \ldots\} \\ &\qquad \text{and for all } i_1, \ldots, i_t \in \{c, c+1, \ldots, s\} \text{ for which} \\ &\qquad P(I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1) \neq 0 \quad \text{and} \\ &\qquad P(I_{t-1} = i_{t-1}) \neq 0. \end{aligned} \tag{2}$$

We start with the definition of conditional probability and (1) and then use assumption (iv).

For all $t \in \{2, 3, \ldots\}$ and for all $i_1, \ldots, i_t \in \{c, c+1, \ldots, s\}$ for which $P(I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1) \neq 0$ and $P(I_{t-1} = i_{t-1}) \neq 0$:

$$P(I_t = i_t \,|I_{t-1} = i_{t-1} \cap I_{t-2} = i_{t-2} \cap \cdots \cap I_1 = i_1)$$

$$= \frac{P(\min\{s, \max\{I_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1)}{P(I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1)} \tag{3}$$

$$= \frac{P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1)}{P(I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1)} \tag{4}$$

$$= P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t). \tag{5}$$

Similarly, for all $t \in \{2, 3, \ldots\}$ and for all $i_1, \ldots, i_t \in \{c, c+1, \ldots, s\}$ for which $P(I_{t-1} = i_{t-1} \cap \cdots \cap I_1 = i_1) \neq 0$ and $P(I_{t-1} = i_{t-1}) \neq 0$:

$$P(I_t = i_t \,|I_{t-1} = i_{t-1})$$

$$= \frac{P(\min\{s, \max\{I_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1})}{P(I_{t-1} = i_{t-1})} \tag{6}$$

$$= \frac{P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1})}{P(I_{t-1} = i_{t-1})} \tag{7}$$

$$= P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t). \tag{8}$$

Combination of (5) and (8) yields (2), which completes the proof. □

Theorem 2. *The element at row i and column j of the transition matrix $\boldsymbol{P}$ of Markov chain $I_1, I_2, I_3, \ldots$ with states $c, c+1, \ldots, s$ is*

$$\begin{aligned} p_{ij} = \sum_{k=0}^{\infty} f_D(k)\, \delta\, (\min\{s, \max\{c - 1 + i - k, 0\} + c\} \\ - c + 1 - j), \quad \text{for all } i, j \in \{1, 2, \ldots, s - c + 1\} \end{aligned} \tag{9}$$

with $\delta(x) = 1$ if $x = 1$ and $\delta(x) = 0$ if $x \neq 1$ for every integer x.

Proof. For all $i, j \in \{1, 2, \ldots, s - c + 1\}$ and for all $t \in \{2, 3, \ldots\}$ for which $P(I_{t-1} = c - 1 + i) \neq 0$:

$$p_{ij} = P(I_t = c - 1 + j\,|I_{t-1} = c - 1 + i) \tag{10}$$

$$= P(\min\{s, \max\{c - 1 + i - D_{t-1}, 0\} + c\} = c - 1 + j) \tag{11}$$

$$\begin{aligned} = \sum_{k=0}^{\infty} f_D(k) \delta\, (\min\{s, \max\{c - 1 + i - k, 0\} \\ + c\} - c + 1 - j). \end{aligned} \tag{12}$$

For getting (10) we applied the definition of the transition matrix and for getting (11) we used (8). □

Theorem 3. *– If $f_D(c) \neq 1$, then for the Markov chain $I_1, I_2, I_3, \ldots$ the following matrix equation in the variable $[f_I(c) f_I(c+1) \cdots f_I(s)]^T$, with $0 \le f_I(c) \le 1$, $0 \le f_I(c+1) \le 1, \ldots$ and $0 \le f_I(s) \le 1$, has a unique solution*

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ p_{12} & p_{22} - 1 & p_{32} & \cdots & p_{s-c+1\ 2} \\ p_{13} & p_{23} & p_{33} - 1 & \cdots & p_{s-c+1\ 3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{1\ s-c+1} & p_{2\ s-c+1} & p_{3\ s-c+1} & \cdots & p_{s-c+1\ s-c+1} - 1 \end{bmatrix} \times \begin{bmatrix} f_I(c) \\ f_I(c+1) \\ f_I(c+2) \\ \cdots \\ f_I(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \tag{13}$$

and for these $f_I(c), f_I(c+1), \ldots$ and $f_I(s)$

$$\lim_{n\to\infty} f_{In}(i) = f_I(i), \quad \text{for all } i \in \{c, c+1, \ldots, s\}. \tag{14}$$

– If $f_D(c) = 1$, then $f_{It}(i) = f_{I1}(i)$ for all $t \in \{1, 2, \ldots\}$ and for all $i \in \{c, c+1, \ldots, s\}$.

Proof. The transition matrix of the Markov chain is

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\ s-c+1} \\ p_{21} & p_{22} & \cdots & p_{2\ s-c+1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{s-c-1\ 1} & p_{s-c-1\ 2} & \cdots & p_{s-c+1\ s-c+1} \end{bmatrix}. \tag{15}$$

Because of Theorem 2, we get Eq. (16) in Box I.

Case 1: $f_D(c) \neq 1$ and $f_D(x) = 0$ for all $x \in \{0, 1, \ldots, c - 1\}$.

By studying (16) we conclude that $\boldsymbol{P}$ is a lower triangular matrix and for every state the probability of going to state c in a number of steps is positive and the probability of going from state i to state j in a number of steps is zero if $i < j$. Therefore state c is recurrent and the other states are transient.

Case 2: $f_D(c) \neq 1$ and $f_D(x) = 0$ for all $x \in \{c+1, c+2, \ldots\}$.

By studying (16) we conclude that $\boldsymbol{P}$ is an upper triangular matrix and for every state the probability of going to state s in a number of steps is positive and the probability of going from state i to state j in a number of steps is zero if $j < i$. Therefore state s is recurrent and the other states are transient.