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Asymptotics of the area under the graph of a Lévy-driven workload process

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ABSTRACT

Let $(Q_t)_{t\in\mathbb{R}}$ be the stationary workload process of a Lévy-driven queue, where the driving Lévy process is light-tailed. For various functions $T(u)$, we analyze

$$\mathbb{P}\left(\int_0^{T(u)} Q_s \mathrm{d}s > u\right)$$

for u large. For $T(u) = o(\sqrt{u})$ the asymptotics resemble those of the steady-state workload being larger than $u/T(u)$. If $T(u)$ is proportional to $\sqrt{u}$ they look like $e^{-\alpha\sqrt{u}}$ for some $\alpha > 0$. Interestingly, the asymptotics are still valid when $\sqrt{u} = o(T(u))$, $T(u) = o(u)$, and $T(u) = \beta u$ for β suitably small.

1. Introduction

Let $(X_t)_{t\in\mathbb{R}}$ be a two sided (one-dimensional) Lévy process with $X_0 = 0$. It is commonly known that under the stability condition $\mathbb{E}X_1 < 0$ the stationary workload process, given by $Q_t := \sup_{s\in(-\infty,t]}(X_t - X_s)$, is well-defined. It is clear that

$$\frac{1}{T}\int_0^T Q_s \mathrm{d}s \tag{1}$$

converges to the steady-state mean workload $\mathbb{E}Q_0$ as $T \to \infty$, as a direct consequence of the ergodic theorem.

However, so far, hardly any explicit results are available on the random variable (1). The objective of the present paper is to study the large-deviation probabilities

$$\pi_{T(u)}(u) := \mathbb{P}\left(\int_0^{T(u)} Q_s \mathrm{d}s > u\right)$$

for u large and various types of functions $T(u)$.

This study exhibits an important example—one could even argue that this is actually the most fundamental example that would come to the mind to a queueing theorist—for which the standard Donsker–Varadhan large deviation theory is *not* directly applicable, even under natural assumptions. For instance, as we shall see, if $T(u) = u\beta$, with $\beta < 1/\mathbb{E}Q_0$, then the asymptotics for $\pi_{T(u)}(u)$ are subexponential in u despite assuming that $(X_t)_{t\in\mathbb{R}}$ is a well-behaved, light-tailed Lévy process. In this paper we study the impact of the function $T(\cdot)$ on the tail asymptotics of (1).

More specifically, under natural large deviation conditions, we obtain the following results:

- For $T(u) = o(\sqrt{u})$ the asymptotics resemble those of the steady-state workload being larger than $u/T(u)$:

  $$\log \pi_{T(u)}(u) \sim \log \mathbb{P}\left(Q_0 > \frac{u}{T(u)}\right).$$

  Intuitively this result means that in order to ensure that the area is larger than u, essentially it is just required that Q_0 is larger than $u/T(u)$.
- For $T(u)$ of the form $T\sqrt{u}$, it is shown that

  $$\lim_{u\to\infty} \frac{1}{\sqrt{u}} \log \pi_{T(u)}(u) = -\alpha, \tag{2}$$

  for a constant $\alpha > 0$ that is identified explicitly. For T below an explicitly given threshold the most likely way in which the rare event happens is such that, with overwhelming probability,

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the buffer never idles in $[0, T\sqrt{u}]$; for T above the threshold the most likely path consists essentially of a single 'big' busy period.
- Finally, it is shown that (2) remains valid in the case $\sqrt{u} = o(T(u))$ and $T(u) = o(u)$, and also if $T(u) = \beta u$ for $\beta < 1/\mathbb{E}Q_0$.

The special case of $(X_t)_{t\in\mathbb{R}}$ corresponding to the Brownian motion was already covered in [1]. Many steps in the analysis presented in [1] used explicit properties of the Brownian motion that are not available for light-tailed Lévy processes. Indeed, one of the challenges of the present paper was to find their counterparts for our more general light-tailed Lévy setting. In addition, the case of $T(u) = \beta u$ with $\beta < 1/\mathbb{E}Q_0$ was not covered in [1]. Also the paper [4] is strongly related to ours. There $\pi_{T(u)}(u)$ is analyzed for the number of customers in the M/M/1 queue, with $T(u) = \beta u$ with $\beta < 1/\mathbb{E}Q_0$. At the methodological level, the analysis presented in this paper borrows elements from both [1,4].

Our paper is organized as follows. In Section 2 the main objective is to find the asymptotics for the auxiliary object

$$\varrho_t := \mathbb{P}\left(\int_0^t X_s \mathrm{d}s/t^2 > a\right); \tag{3}$$

this result, which is extensively used later in the paper, relies precisely on Assumption 1.

Then Section 3 covers the case that $T(u)$ is small relative to $\sqrt{u}$; the analysis relies on straightforward bounds in combination with the classical bound $\mathbb{P}(Q_0 > u) \leq e^{-\kappa u}$ [2]. In Section 4, $T(u)$ is taken proportional to $\sqrt{u}$. Under the assumption that a sample-path large deviation principle is valid, (2) is established. Section 5 covers the cases (i) $\sqrt{u} = o(T(u))$ and $T(u) = o(u)$, and (ii) $T(u) = \beta u$ with $\beta < 1/\mathbb{E}Q_0$. In [4] it was shown that the probability that the sum of γu Weibullian random variables, each of them behaving as $e^{-\alpha\sqrt{u}}$, exceeding u, essentially behaves as a *single* of those Weibullians exceeding u (in terms of logarithmic asymptotics). Relying on this property it is shown that (2) is valid in this case as well. Finally, in Section 6 we discuss how to relax some of the simplifying assumptions that we imposed, and how to obtain results assuming non-stationary initial conditions. Appendix provides some technical large deviation results.

2. Asymptotics of the integral of a Lévy process

The main objective of this section is to study large deviation asymptotics for the auxiliary quantity ϱ_t defined in (3). In our future analysis of $\pi_{T(u)}(u)$, which is the main goal of the paper, only logarithmic results for ϱ_t might suffice; however, because sharp analysis is not difficult to perform and ϱ_t is of independent interest we provide exact asymptotics.

First, define $\phi(\vartheta) := \log \mathbb{E}e^{\vartheta X_1}$ and let $Z(t) := t^{-1}\int_0^t X_s \mathrm{d}s$. Introducing the notation $\eta_t(\vartheta) := \mathbb{E}\exp(\vartheta Z(t))$ and applying integration by parts, we obtain

$$\int_0^t X_s \mathrm{d}s = tX_t - \int_0^t s\mathrm{d}X_s = \int_0^t (t-s)\,\mathrm{d}X_s,$$

to conclude that

$$\eta_t(\vartheta) = \exp\left(\int_0^t \phi\left(\vartheta\left(\frac{t-s}{t}\right)\right)\mathrm{d}s\right) = \exp\left(t\int_0^1 \phi(\vartheta u)\,\mathrm{d}u\right).$$

Consequently, we have that

$$\chi_t(\vartheta) := \frac{1}{t}\log \eta_t(\vartheta) = \int_0^1 \phi(\vartheta u)\,du. \tag{4}$$

The quantities $\chi_t(\vartheta)$ and $\eta_t(\vartheta)$ play an important role in the characterization of the exact asymptotics of ϱ_t. In order to develop such asymptotics, we shall impose throughout the rest of the paper the following assumption, which is fairly standard when developing large deviation estimates.

Assumption 1 (*Steepness to the Right*)**.** If $\phi(\theta) = \log(\mathbb{E}e^{\theta X_1})$, then for every $a > \mathbb{E}X_1$ there exists $\theta^* > 0$ such that $\phi'(\theta^*) = a$.

Using expression (4) we can obtain logarithmic asymptotics (via an application, for instance, of the Gärtner–Ellis theorem, see [7]) leading to

$$\lim_{t\to\infty}\frac{1}{t}\log\varrho_t = -\sup_{\vartheta\geq 0}\left(a\vartheta - \int_0^1 \phi(\vartheta x)\mathrm{d}x\right) =: -J(a). \tag{5}$$

The convexity of $\phi(\cdot)$ implies the convexity of $\vartheta \mapsto \int_0^1 \phi(\vartheta x)\,\mathrm{d}x$, which in turn implies (together with the fact that $\phi(0) = 0$, and that $a > \phi'(0)/2$) that the supremum in (5) is the same if one optimizes over $\vartheta \in \mathbb{R}$. Moreover, it also follows that conditions for local optimality imply global optimality in the optimization problem underlying the definition of $J(a)$. The next lemma, therefore, shows that there is a unique optimizer $\vartheta^\star$ to the previous optimization problem.

Lemma 1. *For every $a > \mathbb{E}Y/2$, there exists $\vartheta^\star > 0$ such that $a = \int_0^1 \phi'(\vartheta^\star x)x\,\mathrm{d}x$.*

Proof. Because of the monotone convergence theorem, it follows that for $\vartheta \in (0, \theta^*)$,

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\int_0^1 \phi(\vartheta x)\mathrm{d}x = \int_0^1 \phi'(\vartheta x)x\,\mathrm{d}x,$$

where $\phi'(\cdot)$ is the derivative of $\phi(\cdot)$. In turn, we have that

$$\int_0^1 \phi'(\vartheta x)\mathrm{d}x = \frac{1}{\vartheta^2}\int_0^\vartheta \phi'(y)y\,\mathrm{d}y.$$

We must show that there exists a unique solution to the equation

$$\vartheta^2\left(a - \phi'(0)/2\right) = \int_0^\vartheta (\phi'(y) - \phi'(0))y\,\mathrm{d}y. \tag{6}$$

By the strict convexity of ϕ, which follows because Y is a non-degenerate random variable in view of Assumption 1, we have that $\phi'(y) - \phi'(0) > 0$ if $y > 0$. Moreover, both the right-hand side and the left-hand side of Eq. (6) are convex functions of ϑ. The derivative of the left-hand side is larger than the derivative of the right-hand side for values of ϑ that are sufficiently close to zero, but because of Assumption 1 eventually the derivative of the right-hand side increases superlinearly in ϑ. So, eventually the right hand side overtakes the left-hand side. By continuity, thus, there exists a solution to Eq. (6). The solution must be unique because the optimization problem underlying the definition of $J(a)$ is a strictly concave program. □

Now we are ready to sharpen the large deviation result (5).

Theorem 1. *Suppose, in addition to Assumption 1, that $\phi(\theta) < \infty$ for θ in a neighborhood of the origin. Define*

$$\sigma^2 := \int_0^1 \phi''(\vartheta^\star x)x^2\,\mathrm{d}x.$$

Then, as $t \to \infty$,

$$\varrho_t := \mathbb{P}(Z(t) > at) \sim \frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}\vartheta^\star\sigma}e^{-tJ(a)}.$$