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An improved upper bound for the TSP in cubic 3-edge-connected graphs

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Abstract

We consider the travelling salesman problem (TSP) problem on (the metric completion of) 3-edge-connected cubic graphs. These graphs are interesting because of the connection between their optimal solutions and the subtour elimination LP relaxation. Our main result is an approximation algorithm better than the 3/2-approximation algorithm for TSP in general.

Keywords: Approximation algorithms; Travelling salesman problem; Graphs; Regular graphs

1. Introduction

In this paper we consider the well-known *travelling salesman problem* (*TSP*). Given a complete undirected graph $G_c = (V, E_c)$ with nonnegative edge weights, the goal is to find a shortest (according to the weights) closed tour visiting each vertex exactly once. A closed tour visiting each vertex exactly once is also known as a *Hamiltonian cycle*.

In general the TSP problem is NP-complete and not approximable to any constant. Even the metric variant of TSP, where the edge weights of G_c form a metric over the vertex set, is known to be MAXSNP-hard [8]. However, Christofides [5] introduced an elegant approximation algorithm for metric TSP that produces a solution with an approximation factor 3/2. Almost three decades later, there is still no approximation algorithm with a better factor than Christofides' 3/2 factor. A detailed survey of the results and the history of TSP appears in [7,6].

Let G be an arbitrary graph with edge weights that do not violate the triangle inequality. A *metric completion* on G is the completion of G into a complete graph G_c such that each added edge (u, v) is weighted in a manner so that G_c forms a metric. A *shortest path metric completion* is a metric completion for which

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each edge added is assigned the weight of the shortest path between u and v. Since we only consider shortest path metric completions in this paper, we will refer to them as metric completions. In general, in this paper we study the case when G is a cubic 3-edge connected graph and G_{c} is its metric completion. All edge weights of the original graph are 1.

One of the classical integer programming formulations of the TSP problem is as follows:

$$\min \sum_{(u,v)\in E_{\mathrm{c}}} w_{(u,v)} x_{(u,v)} \tag{1}$$

$$\sum_{(u,v)\in E_{\mathrm{c}}} x_{(u,v)} = 2 \quad \text{for all } v \in V, \tag{2}$$

$$\sum_{u\in S, v\in V\setminus S} x_{(u,v)} \geqslant 2 \quad \text{for all } \emptyset \neq S \subset V, \tag{3}$$

$$x_{(u,v)} \in \{0,1\} \quad \text{for all } u, v \in V. \tag{4}$$

We denote optimal solutions of the above integer program with IP^* and its value with ρ_{IP^*}. As mentioned above, finding IP^* is NP-complete. Changing requirement (4) to require that $x_{(u,v)} \geqslant 0$ yields a linear programming relaxation of the integer program also known as the *subtour elimination relaxation* (SER for short).

Despite the drawback that there are an exponential number of constraints (of type (3)), SER can nevertheless be solved in polynomial time since each iteration of the separation oracle is one mincut computation. We denote with LP^* the solution to SER and with ρ_{LP^*} its value. A measure for the quality of approximation of an SER solution is the worst-case ratio (integrality gap)

$$\rho = \max \frac{\rho_{APX}}{\rho_{LP^*}},$$

where ρ_{APX} is a value of an SER solution and the maximum is over all instances of metric TSP.

The best-known upper bound to date [10,11] shows that $\rho \leqslant 3/2$. However, no example showing ρ to be $3/2$ is known. The well-known, and currently best, lower bound on ρ appears in Fig. 1. The full example is a metric completion on this graph. To see the $4/3$ gap, set $x_{(u,v)}$ to be $1/2$ for the weight 2 edges in the graph and $x_{(u,v)}$ to be 1 for the weight 1 edges. For all other edges, set $x_{(u,v)} = 0$. In Fig. 1 the optimal tour is shown in the right-hand figure.

This leads to the following conjecture with regard to ρ.

Conjecture. *For metric TSP, the integrality gap ρ for SER is* $4/3$.

There have been numerous attempts to prove this conjecture true; however, so far these attempts have been unsuccessful. In fact, not only has the conjecture not been proven so far, there has not even been an improvement over the $3/2$ factor. This has raised the counterquestion: perhaps $3/2$ is the best factor achievable for the metric TSP problem. Or a somewhat milder variant of the question is, perhaps for $\rho = \rho_{IP^*}/\rho_{LP^*}$, $\rho = 3/2$ is the best achievable.

An optimal solution to the subtour elimination LP is closely related to k-regular k-edge-connected multigraphs in the following sense. Given an optimal solution to the LP, we define the number D to be the smallest common multiplier of the variables x_e^* for all edges e, i.e. D satisfies that for each edge e, Dx_e^* is integer. On the vertex set V we define a multigraph with Dx_e^* edges between vertices u and v where $e = (u, v)$. This multigraph is $2D$-regular and $2D$-edge-connected by the constraints of the LP. Consider the original graph G, which was later completed to a metric as a TSP instance. Say the weights of all edges were 1, i.e. it was unweighted. If the optimal value of the LP was n, i.e. G was "fractionally" Hamiltonian, then it would be enough to find a tour of value αn with $\alpha < 3/2$ to improve on Cristofides algorithm for that special (yet very general) case.

We consider probably the simplest class of graphs satisfying the above properties, the cubic (that is, 3-regular) 3-edge-connected simple graphs with weights 1. The metric completions of these graphs have weights that can be substantially large. It is easily seen that $\rho_{LP^*} = n$ on this class of graphs. Indeed, consider the following solution: $x_{(u,v)} = 2/3$ if $(u, v) \in E$ and $x_{(u,v)} = 0$ otherwise. The feasibility of this solution follows from the fact that graph G is cubic and 3-edge-connected. Since the value of this solution is n and n is a trivial lower bound on ρ_{LP^*}, this solution is optimal.

Thus, to improve the bound on the value of the optimal Hamiltonian cycle, it is enough to prove that there is a Hamiltonian cycle of weight at most αn in G_{c} for $\alpha < 3/2$. In this paper, we design a polynomial