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Reduced-load equivalence for Gaussian processes

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Abstract

We consider a fluid model fed by two Gaussian processes. We obtain necessary and sufficient conditions for the workload asymptotics to be completely determined by one of the two processes, and apply these results to the case of two fractional Brownian motions.

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1. Introduction

Consider two independent stochastic processes $\{X(t), t \geqslant 0\}$ and $\{Y(t), t \geqslant 0\}$. An important problem in applied probability is to determine the behavior of

$$\mathbb{P}\left\{\sup_{t\geqslant 0}[X(t)+Y(t)-ct]>u\right\}, \quad u\to\infty. \tag{1}$$

Applications arise in queueing theory, where (1) represents an overflow probability of a fluid queue fed by two input sources X and Y, and insurance, where (1) can be interpreted as a ruin probability of an insurance company with total premium rate c, initial capital u and claim processes X and Y.

In the latter context, an important class of risk models are 'perturbed' risk models. In this case the principal stream of claims (say Y) is perturbed by a second stream of smaller claims (say X). Perturbed risk models have recently received quite some attention in the literature and are reviewed by Schmidli [17].

A key issue is the influence of the noise process X on the ruin probability. This is the subject of the present paper. In particular, we investigate whether the process X has influence at all when the initial capital u is large. In mathematical terms, this question can be

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rephrased as follows. Assume that X is centered such that it has mean 0. Then the question we investigate is under what conditions

$$\mathbb{P}\left\{\sup_{t\geqslant 0}[X(t)+Y(t)-ct]>u\right\} \sim \mathbb{P}\left\{\sup_{t\geqslant 0}[Y(t)-ct]>u\right\}, \quad u\to\infty. \tag{2}$$

This type of question has been investigated extensively for fluid queues with on–off input, see e.g. [1,12,13], and has also received some attention in the case of classical risk models (see [17]). The property (2) is commonly referred to as a *reduced-load equivalence* (RLE). The goal of the present paper is to complement the above-mentioned results with a treatment of the case where X and Y are *Gaussian* processes. Such processes have become important tools in the performance analysis of communication systems because of their parsimonious nature and the fact that they serve as approximations for the traffic from the superposition of a large number of on–off sources, as is demonstrated in e.g. [6,14]. Recent papers revealing the engineering relevance of Gaussian processes are Choe and Shroff [3] and Wischik [18]. These papers also contain further useful references.

The main result of this paper provides necessary and sufficient conditions for (2) to hold in a Gaussian setting. These conditions are stated in Theorems 2 and 3, respectively. When both X and Y are fractional Brownian motions (FBMs) with Hurst parameters H_X and H_Y, respectively, these theorems imply that (2) is valid when $H_Y > \frac{1}{2}+\frac{1}{2}H_X$, and (2) is not valid when the reverse inequality holds.

At a heuristic level, our results can be explained as follows. The process $X(t)+Y(t)-ct$ is well-known to reach a large value u most likely before time lu, with l a sufficiently large constant. At that time, the deviation of $\sup_{s<t} X(s)$ from its mean is of the order $\sigma_X(lu)=\mathrm{O}(\sigma_X(u))$, with $\sigma_X(u)$ the standard deviation of $X(u)$. Thus, we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\geqslant 0}[X(t)+Y(t)-ct]>u\right\} &\approx \mathbb{P}\left\{\sup_{0<t<lu}[X(t)+Y(t)-ct]>u\right\}\\ &\approx \mathbb{P}\left\{\sup_{0<t<lu}[Y(t)-ct]+\sup_{0<t<lu}X(t)>u\right\}\\ &\approx \mathbb{P}\left\{\sup_{t\geqslant 0}[Y(t)-ct]>u-\mathrm{O}(\sigma_X(u))\right\}.\end{aligned}$$

This heuristic computation indicates that (2) holds if

$$\mathbb{P}\left\{\sup_{t\geqslant 0}[Y(t)-ct]>u\right\} \sim \mathbb{P}\left\{\sup_{t\geqslant 0}[Y(t)-ct]>u-\sigma_X(u)\right\}. \tag{3}$$

As is shown in Theorem 2, this condition is indeed sufficient for (2) to hold. A similar condition has been identified earlier by Jelenković et al. [13] in the case where $Y(t)$ is an on–off source and $X(t)$ is a regenerative process satisfying some regularity conditions, implying $\sigma_X(u)\sim \text{const}\sqrt{u}$. The proof of our main result is similar to the one in [13]. In particular, we almost fully rely on simple sample-path bounds, avoiding more complicated techniques like the double-sum method in Piterbarg [16]. Checking the condition (3) requires information about the exact tail asymptotics of $\mathbb{P}\{\sup_{t\geqslant 0}[Y(t)-ct]>u\}$ as $u\to\infty$. These asymptotics are available when Y is a fractional Brownian motion, see Hüsler and Piterbarg [11]. Further work in this direction has recently been pursued by Dieker [9]. Both papers also contain results for some non-stationary Gaussian processes.

The remainder of the paper is organized as follows. In Section 2 we introduce some notation and state a few preliminary results. In Section 3 we present our main results and discuss some of their implications. Proof details are provided in Section 4.

2. Notation and preliminaries

In this section we introduce some notation and state a few preliminary results. We consider a fluid queue with infinite buffer size and constant drain rate c fed by two independent traffic processes X and Y having stationary increments. Denote by $X(t)$ and $Y(t)$ the amount of traffic generated by the two processes during the time interval $[-t,0]$. If $\mathbb{E}\{X(1)+Y(1)\}<c$,