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Average fill rate and horizon length

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Abstract

Given a sequence of independent and identically distributed demands and an order up to replenishment policy with negligible lead time, we prove that average fill rate is monotonically decreasing in the number of periods in the planning horizon. This was conjectured to be true in a recent issue of this journal.

Keywords: Inventory theory; Finite horizon; Fill rate

1. Introduction

Consider a simple inventory system for a single product over a finite horizon. Demand is represented by a sequence of positive valued iid random variables. There are as many terms in the sequence as there are periods in the horizon. The replenishment system is of the order up to type. In a recent issue of this journal, Chen et al. [1] proved the following result for the system just described: *expected fill rate over a finite horizon with two or more periods is smaller than expected fill rate over a single period and greater than expected fill rate over an infinite horizon, assuming negligible lead time.* As pointed out by the authors in [1], this result has the interesting implication that the customary formula

$$\text{Fill rate} = \frac{\text{Average Number of Units of Demand Filled}}{\text{Average Demand}},$$

which applies exactly to periodic review inventory systems over an infinite horizon, underestimates the fill rate achieved over a finite horizon.

It is conjectured in [1] that for a fixed order up to level, expected fill rate over a finite horizon is a monotonically decreasing function of the number of periods in the horizon; in this paper, we prove the conjecture. Incidentally, all the results about mean fill rate

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obtained by the authors in [1] follow from the proof of the conjecture (Theorem 2 of the present paper).

2. Results and discussion

Let $X_1, X_2, \ldots, X_i, \ldots$ denote an iid sequence of positive valued demand random variables. Let s, a fixed positive number, denote the order up to level. Then, provided that the replenishment lead time is zero, the number of units of demand satisfied in period i is $\mathrm{Min}[X_i, s]$. We write $Y_i = \mathrm{Min}[X_i, s]$. Then expected fill rate over k periods is

$$E\left[\frac{Y_1 + \cdots + Y_k}{X_1 + \cdots + X_k}\right].$$

Chen, Lin and Thomas proved the following theorem:

Theorem 1. *Let i be any positive integer and let k be any positive integer greater than* 1. *Then*

$$E\left[\frac{Y_i}{X_i}\right] \geqslant E\left[\frac{Y_1 + \cdots + Y_k}{X_1 + \cdots + X_k}\right] \geqslant \lim_{j\to\infty} E\left[\frac{Y_1 + \cdots + Y_j}{X_1 + \cdots + X_j}\right].$$

We prove a stronger result, which was conjectured to be true by the authors of Theorem 1.

Theorem 2.

$$E\left[\frac{Y_1 + \cdots + Y_k}{X_1 + \cdots + X_k}\right]$$

is non-increasing in k.

Notice that Theorem 1 can be deduced immediately from Theorem 2. Further, Theorem 2 validates the tacit assumption made by Chen et al. [1] that the sequence

$$u_k = E\left[\frac{Y_1 + \cdots + Y_k}{X_1 + \cdots + X_k}\right]$$

has the property that $\liminf u_k = \limsup u_k$ (that is, the sequence has no limit points other than the unique limit). It follows from Theorem 2 that for a fixed demand distribution and an order up to replenishment policy with negligible lead time, the conventional formula for fill rate is a progressively better approximation to the actual mean fill rate for finite horizon inventory systems spanning a greater number of periods.

The related inequality

$$E\left[\frac{k}{X_1 + \cdots + X_k}\right] \geqslant E\left[\frac{k+1}{X_1 + \cdots + X_{k+1}}\right]$$

follows from the fact that if $X_1, \ldots, X_k, X_{k+1}$ are positive valued iid random variables, then

$$\frac{X_1 + \cdots + X_k}{k}$$

is greater than

$$\frac{X_1 + \cdots + X_k + X_{k+1}}{k+1}$$

in the convex order ([3], Theorem 2.A. 12). It is interesting to note that Theorem 2 asserts that the numbers k and $k+1$ in the numerators of the above inequality can be replaced by the random variables $Y_1 + \cdots + Y_k$ and $Y_1 + \cdots + Y_k + Y_{k+1}$ respectively, despite the stochastic dependence between Y_i and X_i in the operand of the expectation operator. It is natural to enquire whether Theorem 2 can be extended to a class of functions $h(X_i)$ that includes $\mathrm{Min}(X_i, s)$ as a special case. The class of increasing concave functions would seem to be a promising candidate but it is ruled out by the following counterexample: if $Y_i = X_i - s (s > 0)$, then

$$E\left[\frac{Y_1 + \cdots + Y_k}{X_1 + \cdots + X_k}\right] \leqslant E\left[\frac{Y_1 + \cdots + Y_{k+1}}{X_1 + \cdots + X_{k+1}}\right]$$

(this follows from the convex ordering result alluded to, and a little algebraic manipulation).

3. Proof of Theorem 2

We prove the theorem for arbitrary positive valued discrete random variables with finite support (that is, for positive valued simple random variables). The result extends to arbitrary positive valued random variables by convergence: for every positive valued random variable X, there exists a sequence of positive valued simple random variables converging to X pointwise. Let s be a fixed strictly positive real number. Let the underlying distribution consist of the points $a_1, \ldots, a_u$ and $b_1, \ldots, b_v$ where $0 \leqslant a_1 < a_2 < \cdots < a_u < s < b_1 < b_2 < \cdots < b_v$. Further, suppose the distribution attaches probability p_i to the point a_i $(i = 1, \ldots, u)$ and probability q_i to the point b_i $(i = 1, \ldots, v)$, so that $\sum_{i=1}^{u} p_i + \sum_{i=1}^{v} q_i = 1$.