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A skewed Kalman filter

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Abstract

The popularity of state-space models comes from their flexibilities and the large variety of applications they have been applied to. For multivariate cases, the assumption of normality is very prevalent in the research on Kalman filters. To increase the applicability of the Kalman filter to a wider range of distributions, we propose a new way to introduce skewness to state-space models without losing the computational advantages of the Kalman filter operations. The skewness comes from the extension of the multivariate normal distribution to the closed skew-normal distribution. To illustrate the applicability of such an extension, we present two specific state-space models for which the Kalman filtering operations are carefully described.

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1. Introduction

The overwhelming assumption of normality in the Kalman filter literature can be understood for many reasons. A major one is that the multivariate distribution is completely characterized by its first two moments. In addition, the stability of the multivariate normal distribution under summation and conditioning offers tractability and simplicity. Therefore, the Kalman filter operations can be performed rapidly and efficiently whenever the

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normality assumption holds. However, this assumption is not satisfied for a large number of applications. For example, some distributions used in a state-space model can be skewed. In this work, we propose a novel extension of the Kalman filter by working with a larger class of distributions than the normal distribution. This class is called *closed skew-normal distributions*. Besides introducing skewness to the normal distribution, it has the advantages of being closed under marginalization and conditioning. This class has been introduced by González-Farías et al. [9] and is an extension of the multivariate skew-normal distribution first proposed by Azzalini and his coworkers [1–**?**4]. These distributions are particular types of generalized skew-elliptical distributions recently introduced by Genton and Loperfido [8], i.e. they are defined as the product of a multivariate elliptical density with a skewing function.

This paper is organized as follows. In Section 2, we recall the definition of the closed skew-normal distribution and the basic framework of state-space and Kalman filtering. Section 3 presents the conditions under which the observation and state vectors of the state-space model follow closed skew-normal distributions. In Section 4, a sequential procedure based on the Kalman filter is proposed to estimate the parameters of such distributions. A simulated example illustrates the differences between the classical Kalman filter and our non-linear skewed Kalman filter. We discuss our strategy relative to other Kalman filters and conclude in Section 5.

2. Definitions and notations

2.1. The closed skew-normal distribution

The closed skew-normal distribution is a family of distributions including the normal one, but with extra parameters to regulate skewness. It allows for a continuous variation from normality to non-normality, which is useful in many situations, see e.g. Azzalini and Capitanio [4] who emphasized statistical applications for the skew-normal distribution.

An n-dimensional random vector X is said to have a multivariate closed skew-normal distribution [9,10], denoted by $CSN_{n,m}(\mu, \Sigma, D, \nu, \Delta)$, if it has a density function of the form

$$\frac{1}{\Phi_m(0; \nu, \Delta + D\Sigma D^T)} \phi_n(x; \mu, \Sigma)\Phi_m(D(x-\mu); \nu, \Delta), \quad x \in \mathbb{R}^n, \tag{1}$$

where $\mu \in \mathbb{R}^n$, $\nu \in \mathbb{R}^m$, $\Sigma \in \mathbb{R}^{n\times n}$ and $\Delta \in \mathbb{R}^{m\times m}$ are both covariance matrices, $D \in \mathbb{R}^{m\times n}$, $\phi_n(x; \mu, \Sigma)$ and $\Phi_n(x; \mu, \Sigma)$ are the n-dimensional normal pdf and cdf with mean μ and covariance matrix Σ. When $D = 0$, the density (1) reduces to the multivariate normal one, whereas it reduces to Azzalini and Capitanio's [4] density when $m = 1$ and $\nu = D\mu$. The matrix parameter D is referred to as a "shape parameter". The moment generating function $M_{n,m}(t)$ for a *CSN* distribution is given by

$$M_{n,m}(t) = \frac{\Phi_m(D\Sigma t; \nu, \Delta + D\Sigma D^T)}{\Phi_m(0; \nu, \Delta + D\Sigma D^T)} \exp\left\{\mu^T t + \frac{1}{2}(t^T \Sigma t)\right\} \tag{2}$$