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When is the mean self-consistent?

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Abstract

We study the conditions under which the sample mean is self-consistent, and therefore an optimal predictor, for an arbitrary observation in the sample.

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1. Introduction

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a finite sequence of random variables. Let S_n and A_n denote their sum and arithmetic mean, respectively, i.e.,

$$A_n = \frac{1}{n} S_n = \frac{1}{n}(X_1 + \cdots + X_n).$$

Suppose that for all i and values of S_n we have

$$\mathbb{E}[X_i | S_n] = A_n. \qquad (1)$$

The relationship (1) means that the conditional expectation of an observation with respect to the sample sum is equal to the arithmetic mean. Conditioning on the sample sum is equivalent to conditioning on the sample mean because there is $1-1$ correspondence between these quantities. Therefore (1) may also be written as $\mathbb{E}[X_i|A_n] = A_n$. This type of relationship

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has been called self-consistency by Tarpey and Flury [7] and used to approximate one random variable by another (e.g., [6]). Clearly if (1) holds the arithmetic mean is an optimal predictor of X_i under squared error loss. Our goal here is to characterize the random vectors **X** for which the equality (1) is true.

2. Independent identically distributed sequences

We start by investigating (1) assuming that $X_1, \ldots, X_n$ are independent and identically distributed (IID) random variables. First note that

$$A_n = \mathbb{E}[A_n | A_n].$$

Therefore,

$$\begin{aligned} A_n &= \mathbb{E}\left[\frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) | A_n\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i|A_n] = \mathbb{E}[X_i|A_n] \end{aligned}$$

for all $i = 1, \ldots, n$ which establishes (1). The proof is based on the linearity of the expectation operator and the symmetry relationship

$$\mathbb{E}[X_i|A_n] = \mathbb{E}\left[X_j|A_n\right] \quad \text{for all} \quad 1 \leqslant i, j \leqslant n. \tag{2}$$

The relationship (2) may be proved analytically (we do that later in a more general setting). However, it is intuitively true because the information in A_n about X_i and X_j is the same. It is important to note that (1) implies that $\mathbb{E}[X_i|A_n] < \infty$ even if the expectation of X does not exist. The identity (1) is well known and plays an important role in the theory of unbiased estimation, specifically in the context of the Rao–Blackwell theorem [2]. Moreover, (1) may be viewed as a projection of X_i onto the space spanned by the unit vector. This type of representation is related to Hoeffding's decomposition and limit theorems for U-Statistics [8]. In the following we show, that under minimal regularity conditions, independent random variables satisfying (1) must be identically distributed. We start with the following Lemma whose proof may be found in [5]. Note that in the following we use the symbol i both as an index (i.e., X_i) and as the imaginary unit $i = \sqrt{-1}$. The dual use is apparent from the context and should cause no confusion.

Lemma 1. *Let Y and X be random variables where $\mathbb{E}[Y]$ exist. Then $\mathbb{E}[Y|X] = 0$ if and only if $\mathbb{E}\left[Y \exp\,(itX)\right] = 0$ for all $t \in \mathbb{R}$.*

Definition 2. Let $\Psi(t)$ be the characteristic function (CF) of the random variable X, we call $O = \{t|\Psi(t) = 0\}$ the zero characteristic set (ZCS) of X.

Theorem 3. *Let $X_1, \ldots, X_n$ be independent random variables with a finite first moment, having a ZCS with Lebesgue measure zero. If, additionally, they satisfy (1) then they must be identically distributed.*