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Extension of the variance function of a steep exponential family

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Abstract

Let $F = \{P(m, F);\ m \in M_F\}$ be a multidimensional steep natural exponential family parameterized by its domain of the means M_F and let $V_F(m)$ be its variance function. This paper studies the boundary behaviour of V_F. Necessary and sufficient conditions on a point $\overline{m}$ of ∂M_F are given so that V_F admits a continuous extension $V_F(\overline{m})$ to the point $\overline{m}$. It is also shown that the existence of $V_F(\overline{m})$ implies the existence of a limit distribution $P(\overline{m}, F)$ concentrated on an exposed face of $\overline{M}_F$ containing $\overline{m}$. The relation between $V_F(\overline{m})$ and $P(\overline{m}, F)$ is established and some illustrating examples are given.

Keywords: Exponential family; Variance function; Steepness; Convex; Exposed face

1. Introduction

Exponential families have been a distinguished topic of theoretical statistics and probability theory for several decades. As they provide a general framework for many practical optimization problems in statistics, their behaviour on the boundary of natural parameter or mean parameter spaces is of theoretical interest. Our approach in the present work is based on the natural exponential families and their description through means (see [7]). In this context, the variance function of a natural exponential family (NEF) appears as the most appropriate tool, and so it has received a great deal of attention in the statistical literature. Its importance stems from the fact that it characterizes the family within the class of all natural

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exponential families [9,11]. Several classifications of NEFs by means of variance functions have been defined (see, [2,3,8,9]). Also, many characteristic properties of classes of distributions have been established using variance functions (see, [4,6]). Beside its role for the study of NEFs, the variance function itself has many nice intrinsic algebraic properties. Jørgensen [5] asked what properties are shared by all members of a natural exponential family F in terms of its variance function V_F. He considered the one-dimensional version of the problem. In this case, V_F is a real valued function and the domain of the means M_F is an interval of $\mathbb{R}$ which, naturally, has at most two extremities. The problem involves studying the behaviour of V_F at one of the extremities of M_F. We are concerned with multidimensional steep NEFs, that is, NEFs with the domain of their means equal to the interior of the convex hull of the support. This global assumption is satisfied in all reasonable multidimensional cases and it is justified from a technical point of view. The steepness enables to apply the convex analysis methods to the convex supports of NEFs. A natural problem within this approach is to identify the points of the boundary of the domain of the means where the variance function V_F admits an extension. In Section 2, we show that V_F extends continuously to a point on the boundary if and only if V_F is bounded in its neighbourhood. We also show that this is equivalent to the existence of a special bounded neighbourhood of the point. Section 3 is devoted to continuous extensions of the mapping $m \mapsto P(m, F)$ in the weak topology. We prove that boundedness of V_F is sufficient for these extensions and that limit distributions $P(\overline{m}, F)$ are concentrated on faces of the closure $\overline{M}_F$. In this section we also give the link between $V_F(\overline{m})$ and the variance of the limit distribution $P(\overline{m}, F)$. Proofs are postponed to Section 4.

2. Extension of a variance function

We introduce first some notation and review some basic concepts concerning exponential families and their variance functions. For more details, we refer the reader to [7].

For a positive Radon measure μ on $\mathbb{R}^d$, we denote

$$L_\mu : \mathbb{R}^d \to]0, +\infty[: \theta \mapsto \int_{\mathbb{R}^d} \exp\langle \boldsymbol{\theta}, \mathbf{x} \rangle \mu(dx)$$

the Laplace transform, where $\langle \theta, x \rangle$ is the ordinary scalar product of θ and x in $\mathbb{R}^d$. Also we denote

$$\Theta(\mu) = interior\{\theta \in \mathbb{R}^d;\ L_\mu(\theta) < +\infty\},$$

$$k_\mu = \log L_\mu;$$

k_μ is the cumulant generating function of μ.

The set $\mathscr{M}(\mathbb{R}^d)$ is defined as the set of positive measures μ that are not concentrated on an affine hyperplane and $\Theta(\mu)$ is not empty.