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Singular random matrix decompositions: distributions[☆]

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Abstract

Assuming that Y has a singular matrix variate elliptically contoured distribution with respect to the Hausdorff measure, the distributions of several matrices associated to QR, modified QR, SV and polar decompositions of matrix Y are determined, for central and non-central, non-singular and singular cases, as well as their relationship to the Wishart and pseudo-Wishart generalized singular and non-singular distributions. Some of these results are also applied to two particular subfamilies of elliptical distributions, the singular matrix variate normal distribution and the singular matrix variate symmetric Pearson type VII distribution.

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1. Introduction

It has been a common practice in the past, to eliminate variables or individuals to correct for dependencies among columns or rows when we sample from a multivariate distribution. This solution in part, was due to the fact of not having a distribution theory to handle all those cases. In a more formal way, let $Y \in \mathbb{R}^{N\times m}$, be a sample of N individuals with m variables under study, if there exist dependencies among rows (individuals) or columns (variables), Y does not have a density with respect to the Lebesgue measure in $\mathbb{R}^{Nm}$. However, it is known that Y has a density on a subspace $\mathcal{M} \subset \mathbb{R}^{Nm}$ on which it is possible to define a measure called the Hausdorff measure, which coincides with the Lebesgue measure when it is defined on $\mathcal{M}$ (for the cases studied in Section 3, $\mathcal{M}$ is an affine subspace). Details on this kind of problems can be found in [10,12]. They proposed expressions for the singular matrix variate normal distribution and singular matrix variate elliptically contoured distribution. In other words, we count now with a solution for the classical multivariate statistical analysis when based on the normal distributions and also for the more general case called the generalized multivariate statistical analysis based on the elliptically contoured distributions.

When Y has a distribution with respect to the Lebesgue measure we could find different ways of deriving the Wishart distribution. Some are based on the QR decomposition, [29,32,33], others on the singular value decomposition (SVD), [23] and some others on the polar decomposition [2,22]. What all of these approaches through different factorizations are trying to do is to find an alternative coordinate system for the columns (or rows) for the matrix Y. For example, the coordinates obtained from the QR decomposition are called rectangular coordinates, [31, p. 597], for the polar decomposition, polar coordinates [2], etc. These matrices of coordinates, besides of being the key part for establishing the Wishart, pseudo-Wishart, F and beta distributions, as well as distributions of $|Y'Y|$ and $\operatorname{tr} Y'Y$ among others, play an important role in other areas of knowledge, in particular on the shape theory and pattern recognition. As an example, if Y has a matrix variate normal distribution, it may be written as $Y = H_1 T$, the QR decomposition. In the context of shape theory, the distribution of T is called size-and-shape distribution, also known in the literature as the rectangular coordinates distribution, see [20,31, p. 597]. In the same setting of shape theory, when considering the SV ($Y = V_1 D W_1$) or polar ($Y = P_1 R$) decompositions, the matrices (D, W_1) and R may both be thought of as an alternative coordinates system, in such a way that the corresponding distributions play the role of size-and-shape distributions, see [18,28]. Similarly, matrix D is considered as yet another coordinate system, and its corresponding distribution is called size-and-shape cone distribution, see [11,12,20]. Some of these results were extended to the case in which Y has a singular gaussian and a singular elliptically contoured distribution, see [10,11]. In the context of pattern recognition the role of some of these decomposition is also known, in particular the SV decomposition is known as the Karhunen–Lòeve expansion or decomposition [26].

In the present work some results on distributions of random matrices, for which their density function exist with respect to the Lebesgue measure, will be extended to the case in which Y has a density with respect to the Hausdorff measure, and moreover, to the case in which Y has a singular matrix variate elliptically contoured distribution. In Section 2, the densities of matrices associated to the QR, modified QR, SV and polar decompositions are found with respect to the Hausdorff measure, both for the non-central and central cases.