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# Sparse moving maxima models for tail dependence in multivariate financial time series

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ABSTRACT

The multivariate maxima of moving maxima (M4) model has the potential to model both the cross-sectional and temporal tail-dependence for a rich class of multivariate time series. The main difficulty of applying M4 model to real data is due to the estimation of a large number of parameters in the model and the intractability of its joint likelihood. In this paper, we consider a sparse M4 random coefficient model (SM4R), which has a parsimonious number of parameters and it can potentially capture the major stylized facts exhibited by devolatized asset returns found in empirical studies. We study the probabilistic properties of the newly proposed model. Statistical inference can be made based on the Generalized Method of Moments (GMM) approach. We also demonstrate through real data analysis that the SM4R model can be effectively used to improve the estimates of the Value-at-Risk (VaR) for portfolios consisting of multivariate financial returns while ignoring either temporal or cross-sectional tail dependence could potentially result in serious underestimate of market risk.



## 1. Introduction

The most recent global financial crisis manifested by the subprime mortgage crisis and severe crash in stock markets highlights the importance of predicting rare events and boosts up the demand for efficient tools of assessing market risk, e.g., the Value-at-Risk (VaR). In statistical terminology, the VaR of a certain asset or portfolio is the quantile of its negative return distribution corresponding to a given high probability (e.g., McNeil et al., 2005; Tsay, 2005). Due to the strong empirical evidences suggesting heteroscedasticity in financial time series, the VaR conditional on the current volatility background is of more relevance when we are concerned about the short-term potential loss under adverse market environment. Ever since McNeil and Frey (2000), pseudo maximum likelihood and extreme value theory have been incorporated together to estimate the conditional VaR of a given asset where the proposed methodology amounts to estimating the stochastic volatility through fitting an ARMA-GARCH model to the original return series and extrapolating the tail distribution of the unobserved innovations (proxied by the standardized residuals, or devolatized returns) by the Generalized Pareto Distribution (see also Lauridsen, 2000; Laurini and Tawn, 2009). As for the VaR of a portfolio with more than one asset, multivariate extreme value theory can be used to identify and characterize the joint extreme features of all the assets' returns within the tail region (Poon et al., 2004). One basic assumption made by the above-mentioned literature is that the innovations are either independent and identically distributed (IID), or IID after declustering across the time domain. However, empirical findings in financial applications suggest that extreme events tend to be clustered in time and

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hence the validity of the IID assumption is questionable. Besides, in many situations, we are interested in the VaR of cumulative loss over a period of more than one day as the Basel Committee requires the use of 10-day VaR at the 99% confidence level for market risk (Basel Committee on Banking Supervision, 2009). In this case, neither the IID nor the declustering approach is appropriate and a flexible and tractable dynamic model capable of describing both temporal and cross-sectional tail dependence of multivariate time series is needed. This paper aims at developing such models which enhance the usage of GARCH models and more efficient VaR calculations in practice. Our proposed model is a kind of "building block" for other more involved models of asset returns. It will be applied to devolatized financial returns.

Useful models for serial tail-dependence of univariate stationary time series include the moving maxima model studied by Deheuvels (1983) and Hall et al. (2002) and the max-autoregressive moving average model studied by Davis and Resnick (1989, 1993). Smith and Weissman (1996) generalized the (univariate) maxima of moving maxima (M3) model to the multivariate maxima of moving maxima (M4) model defined as

$$X_{td} = \max_{l \geq 0} \max_{-\infty < k < \infty} \alpha_{lkd} Z_{l,t-k}, \quad d = 1, \ldots, D, \ -\infty < t < \infty, \tag{1}$$

where $\{\alpha_{lkd}\}$ are nonnegative constants satisfying $\sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty}\alpha_{lkd} = 1$, $d = 1, \ldots, D$, and $\{Z_{lk}\}$ is an array of IID unit Fréchet random variables with distribution function $e^{-1/z}$, $z > 0$. Smith and Weissman showed that the M4 (M3) model characterizes the extreme behavior of a rich class of multivariate (univariate) stationary time series. Note that model (1) is seldom directly fitted to observed data in financial application; instead, it is mainly used to describe the extreme behaviors within the residuals obtained from a filter, such as the ARMA-GARCH filter employed by McNeil and Frey (2000). As found in Poon et al. (2004), Zhang (2005), and Zhang and Smith (2010), the mean cluster size of exceedances over a high threshold is in general small for devolatized financial return series, which leads us to consider a sparse version of the M4 model as a trade-off between theoretical richness of the M4 process and practical tractability of the sparse specification.

A probabilistic property of an M3 or M4 process given by (1) is the so-called "signature patterns" when $l$ and $k$ vary in finite ranges, e.g., $l = 0,1,\ldots,L$ and $k = 0,1,\ldots,K$; see Zhang and Smith (2004). That is, for $Z_{l^*,t_0}$ of huge magnitude, it dominates the shocks $Z_{lt}$ with $t \approx t_0$ and accordingly $X_{t_0+k,d}/X_{t_0,d} = \alpha_{l^*kd}/\alpha_{l^*0d}$ for small $k$'s, which is unrealistic in some applications. By introducing a sparse M3 model with time-dependent random coefficients (SM3R), we can eliminate the "signature patterns". Furthermore, the associated extreme profiles of the SM3R model are more flexible due to the randomness of the coefficients and suggest a good match with the empirical implications from the literature.

The cross-sectional tail-dependence of the M4 process is described by the loadings $\{\alpha_{lkd}\}$ on a common set of "shock" variables $\{Z_{lk}\}$. For $X_{td}$ and $X_{td'}$ generated from the M4 model, they are tail-dependent as long as they have positive loadings on a common "shock" $Z_{lk}$. We consider a special variation of the original structure for cross-sectional tail-dependence under the M4 model which we believe is adequately flexible and compatible with all kinds of parametric models for multivariate extremes proposed in this field. By incorporating this new structure into the SM3R model, we get the SM4R model which can be viewed as a modified and generalized version of (1).

Estimating the parameters $\{\alpha_{lkd}\}$ is a tough task under the M4 model as the joint distribution function of $(X_{t1}, \ldots, X_{tD})$ is intractable. Zhang and Smith (2010) proposed a procedure based on empirical process while Chamú-Morales (2005) proposed a Monte Carlo method under the Bayesian framework for a class of M3 models. We apply the Generalized Method of Moments (GMM) approach for estimation and inference of the parameters of moving-maxima-type models. This approach is easy to apply and has the advantage of utilizing not only the order but also the magnitude of the extreme observations when compared to the estimators proposed in Zhang (2005) and Zhang and Smith (2010).

The rest of this paper is organized as follows. Sections 2 and 3 describe the specifications for temporal and cross-sectional tail dependence of the SM4R model, respectively. Section 4 focuses on constructing the GMM estimators of parameters with their performance examined in Section 5 through simulation experiments. Real data of three major stock indices are analyzed in Section 6. We conclude with some remarks in Section 7. Proofs, derivations, and additional technical details are presented in the supplement of this paper.

## 2. Temporal dependence structure

To model extreme series, model (1) with $l = 0,1,\ldots,L$ and $k = 0,1,\ldots,K$ is often applied. The M3 model ($D=1$) can be written as

$$X_t = \max[\mathbf{A} \cdot \mathbf{Z}^{(t)}], \quad -\infty < t < \infty, \tag{2}$$

$$\mathbf{A} = \begin{pmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0\,K} \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1\,K} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{L0} & \alpha_{L1} & \cdots & \alpha_{LK} \end{pmatrix}, \quad \mathbf{Z}^{(t)} = \begin{pmatrix} Z_{0t} & Z_{0,t-1} & \cdots & Z_{0,t-K} \\ Z_{1t} & Z_{1,t-1} & \cdots & Z_{1,t-K} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{Lt} & Z_{L,t-1} & \cdots & Z_{L,t-K} \end{pmatrix},$$

where $\mathbf{A} \cdot \mathbf{Z}^{(t)}$ represents the componentwise product of matrices $\mathbf{A}$ and $\mathbf{Z}^{(t)}$ and max $\mathbf{C}$ is the maximum overall elements of the matrix $\mathbf{C}$. Unfortunately, it is very difficult to make inference without any condition on $\mathbf{A}$, because it relies on the estimation of $(L+1)(K+1)$ parameters, a number that can be large in practice.