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Normality testing for a long-memory sequence using the empirical moment generating function

Sucharita Ghosh *

Swiss Federal Research Institute WSL, Statistics Lab, CH-8903 Birmensdorf, Switzerland

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ABSTRACT

Moment generating functions and more generally, integral transforms for goodness-of-fit tests have been in use in the last several decades. Given a set of observations, the empirical transforms are easy to compute, being simply a sample mean, and due to uniqueness properties, these functions can be used for goodness-of-fit tests. This paper focuses on time series observations from a stationary process for which the moment generating function exists and the correlations have long-memory. For long-memory processes, the infinite sum of the correlations diverges and the realizations tend to have spurious trend like patterns where there may be none. Our aim is to use the empirical moment generating function to test the null hypothesis that the marginal distribution is Gaussian. We provide a simple proof of a central limit theorem using ideas from Gaussian subordination models (Taqqu, 1975) and derive critical regions for a graphical test of normality, namely the T_3-plot (Ghosh, 1996). Some simulated and real data examples are used for illustration.

1. Introduction

We are concerned with a stochastic process $\{X_i,\ i \geq 1\}$ where

$$X_i = \mu + \sigma Y_i, \tag{1}$$

$-\infty < \mu < \infty$ and $\sigma > 0$ are unknown constants and $\{Y_i\}$ is a stationary process with $E(Y_i) = 0$, $var(Y_i) = 1$ and covariances

$$cov(Y_i, Y_{i+k}) = \gamma(|k|).$$

We assume long-memory (see Beran, 1994), i.e. the covariances $\gamma(|k|)$ decay slowly (hyperbolically) to zero with increasing lags, i.e.

$$\gamma(|k|) \sim C|k|^{2H-2} \text{ as } |k| \to \infty \quad \text{and} \quad 1/2 < H < 1, \tag{2}$$

with $C > 0$. Equivalently if $\psi(\cdot)$ denotes the spectral density of Y_i then

$$\psi(\lambda) \sim D|\lambda|^{-2(H-1/2)}, \quad \text{as } \lambda \to 0. \tag{3}$$

* Tel.: +41 44 739 2431; fax: +41 44 739 2215.
E-mail address: rita.ghosh@wsl.ch

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Here

$$D = \frac{\sin(\pi(H-1/2))\Gamma(2H)}{2H} C.$$

Long-memory in particular implies nonsummability of the covariances i.e.

$$\sum_{k=-\infty}^{\infty} \gamma(k) = \infty.$$

In addition to this correlation structure, we assume that the marginal moment generating function of X_i

$$m(t) = E(e^{tX_i}), \quad t \in \mathbb{R} \tag{4}$$

exists and is finite for t in an open interval around the origin. All parameters of this process are unknown and our problem is as follows: given a set of observations $X_1, X_2, \ldots, X_n$ satisfying (1) through (4), we wish to test the null hypothesis that the marginal moment generating function of $X_1, X_2, \ldots X_n$ is $m_0(\cdot)$ and more specifically

$$H_0 : m_0(t) = e^{\mu t + \sigma^2 t^2/2}$$

i.e. the marginal distribution is Gaussian with an unknown mean μ and an unknown variance $\sigma^2 > 0$. For classical references to the topic of testing normality for iid data, see D'Agostino and Stephens (1986), Gnanadesikan (1977) and Mardia (1980). Our interest here is in graphical methods and for background information see Atkinson (1985), Chambers et al. (1983), Cleveland (1993), Wilk and Gnanadesikan (1968) and others.

To start with, we consider estimation of the population moment generating function (mgf) of X by its empirical version. The empirical moment generating function (emgf) for a set of observations $X_1, X_2, \ldots, X_n$ is defined as

$$m_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{tX_j}. \tag{5}$$

In what follows, we will restrict the values of the argument t on a compact interval around zero. The emgf is unbiased i.e. for every fixed t, $E(m_n(t)) = m(t)$ but as a stochastic process in t it is not stationary. For instance even for independently and identically distributed (iid) observations, the covariance function

$$k_n(t,s) = cov(m_n(t), m_n(s))$$

is nonstationary, in particular

$$k_n(t,s) = \frac{1}{n}[m(t+s) - m(t)m(s)].$$

when $X_1, X_2, \ldots, X_n$ are iid, various asymptotic results including the central limit theorem concerning the emgf and corresponding derivative processes can be found in Csörgő (1980), Feuerverger and McDunnough (1984) and Feuerverger (1989). As for normality testing using the emgf, quite a large literature exists of which we mention only a few namely, Fang et al. (1998; multivariate normality), Ghosh (1996; univariate normality) and Meintanis (2007; skew normal family of distributions or the SND). The SND (Azzalini, 1985; also see Gupta and Chen, 2001; Gupta and Brown, 2001) includes the normal distribution as a special case and seems promising for modeling various natural processes. Note that our 'error' process Y_i may have been generated from a Gaussian subordination model, i.e. $Y_i = G(Z_i)$ where Z_i is a stationary Gaussian process and G is unknown. This class of stochastic processes includes the Gaussian process as well as various others (Taqqu, 1975). Under the null hypothesis of normality, $G(Z_i) = Z_i$. Ramifications of this transformation model will be considered elsewhere. Other ideas for using the emgf are in Epps et al. (1982), Feuerverger (1989), Ghosh and Beran (2000) and in several papers by Koutrouvelis and collaborators (e.g. Kallioras et al., 2006; Koutrouvelis and Canavos, 1997; Koutrouvelis and Meintanis, 2002; Koutrouvelis et al., 2005). This list of references to emgf based approaches is by no means comprehensive but should lead to additional references in this field.

As for properties of the emgf for stochastic processes, when the Y_i's, $i = 1, 2, \ldots$ form a linear process, say of the form $Y_i = \sum_{j=1}^{\infty} a_{i-j} \epsilon_j$ where the $\epsilon_1, \epsilon_2, \ldots$ are iid innovations having the mgf m_ϵ, the emgf is a natural choice for further investigation of the process. This is so because due to the independence of the innovations, the mgf of Y_i can be written in terms of the mgf of the innovations, i.e. $E(e^{tY_i}) = \prod_{j=1}^{\infty} E(e^{ta_{i-j}\epsilon_j}) = \prod_{j=1}^{\infty} m_\epsilon(ta_{i-j})$, $t \in \mathbb{R}$. In this case, two different estimators of the mgf of Y_i can be suggested, one of which uses the properties of the linear process and the other is the empirical moment generating function. Properties of these estimators are considered in Ghosh (2003) and Ghosh and Beran (2006). For additional related information on empirical processes based on linear processes, see Giraitis and Surgailis (1999) and references therein.

Several authors also consider the empirical characteristic function (ecf); e.g. Feuerverger and McDunnough (1981). The ecf is defined like the emgf, except that t is complex, namely, t is replaced by $\imath t$ where $\imath = \sqrt{-1}$. Its real part is then $c_n(t) = (1/n) \sum_{j=1}^{n} \cos(tX_j)$ and the complex part is $s_n(t) = (1/n) \sum_{j=1}^{n} \sin(tX_j)$. The characteristic function (cf) $c(t) = E(\exp(\imath t))$ exists for all distributions and because of its uniqueness property, it has been a popular tool for constructing goodness-of-fit tests. Examples include Epps and Pulley (1983) who use the difference between the squared norm of ecf and the square of the cf of the normal distribution, Csörgő (1986), who generalizes the method of Murota and