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Partially balanced incomplete block designs with two associate classes ☆

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ABSTRACT

We provide constructions of cyclic 2-class PBIBD's using cyclotomy in finite fields. Our results give theoretical explanations of the two sporadic examples given by Agrawal (1987).

1. Introduction

Following Bose et al. (1954), an incomplete block design is said to be partially balanced with two associate classes if it satisfies the following requirements:

1. The experimental material is divided into b locks of k units each, different treatments being applied to units in the same block.
2. There are $v(>k)$ treatments, each of which occurs in r blocks.
3. A relation of association between any two treatments satisfying the following requirements can be established:
   (a) Two treatments are either first associates or second associates.
   (b) Each treatment has exactly n_i ith associates ($i=1,2$).
   (c) Given any two treatments which are ith associates, the number of treatments common to the jth associate of the first and the kth associate of the second is p^i_{jk} and is independent of the pair of treatments we start with. Also $p^i_{jk}=p^i_{kj}(i,j,k=1,2)$.
4. Two treatments which are ith associates occur together in exactly λ_i blocks ($i=1,2$).

For a proper partially balanced incomplete block design, we require $\lambda_1 \neq \lambda_2$. If $\lambda_1 = \lambda_2$ or if one of the n_i vanishes, the design becomes a balanced incomplete block design (BIBD).

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The numbers $v,r,k,b,n_1,n_2,\lambda_1$, and λ_2 are called parameters of the first kind, whereas the numbers $p^i_{jk}(i,j,k=1,2)$ are called parameters of the second kind.

Let Z_v denote the ring of integers modulo v. A two-class cyclic association scheme is defined, see John et al. (1972), Nandi and Adhikary (1966), and Raghavarao (1971), by taking treatments as the distinct elements of Z_v, and the first associates of $i\in Z_v$ by adding i to the numbers $d_1,d_2,\ldots,d_{n_1}$ modulo v, where $d_1,d_2,\ldots,d_{n_1}$ satisfy the following:

1. $d_i\in Z_v$ and $\forall i\neq j\in Z, d_i\neq d_j$.
2. The multiset of differences, $M=(d_i-d_j|i\neq j\in\{1,\ldots,n_1\})$, where each $d_i\in\{d_1,d_2,\ldots,d_{n_1}\}$ occurs p^1_{11} times and everything else occurs p^2_{11} times. Treatments are either first associates or second associates. Designs resulting in λ_1 and λ_2 differing by 1 are called nearly balanced designs.

Two-class cyclic PBIBD's with the following parameters were obtained by Agrawal (1987):

Parameter set 1: $v=17$, $r=10$, $k=5$, $b=34$, $n_1=n_2=8$, $\lambda_1=3$, $\lambda_2=2$, $p^1_{11}=3$, $p^2_{11}=4$.
Parameter set 2: $v=37$, $r=10$, $k=10$, $b=37$, $n_1=n_2=18$, $\lambda_1=3$, $\lambda_2=2$, $p^1_{11}=8$, $p^2_{11}=9$.

No theoretical explanation was provided in Agrawal (1987) for the aforementioned two examples. In this paper, we provide a few theorems using cyclotomy in finite fields, which produce infinite families of cyclic 2-class PBIBD's. The examples given by Agrawal (1987) are shown to be special cases of our theorems.

2. Preliminaries

If R is a commutative ring with unity and G is a group, we let RG denote the group ring of G over R. When G is multiplicative, each subset $S\subseteq G$ is identified with the group ring element

$$S=\sum_{s\in S}s.$$

For $A=\sum_{g\in G}a_g g$ and $t\in Z$ we define

$$A^{(t)}=\sum_{g\in G}a_g g^t.$$

When G is additive, each subset $S\subseteq G$ is identified with the group ring element

$$S=\sum_{s\in S}x^s.$$

For $A=\sum_{g\in G}a_g x^g$ and $t\in Z$ we define

$$A^{(t)}=\sum_{g\in G}a_g c^{tg}.$$

If G is a group and S is a subset of G of cardinality s, it can be easily seen that in the group ring $ZG, SG=sG$. For $m\in Z$ and the identity 1_G of G, we will denote the group ring element $m1_G$ simply by m.

Let q be a prime power. Let $GF(q)$ denote the finite field with q elements. Let ω be a generating element of $GF(q)^*$, the multiplicative group of all the non-zero elements of $GF(q)$. Let $q=ef+1$ for $e>0, f\neq 1$. Then the cyclotomic classes of order e are defined by

$$C_i:=\{\omega^{es+i}:s=0,\ldots,f-1\}\quad\text{for } i=0,\ldots,e-1.$$

The cyclotomic numbers of order e are defined by

$$(i,j)_e:=\left|\{(x,y):x\in C_i,\ y\in C_j,\ x+1=y\}\right|.$$

It is easy to see that

1. If $q\equiv 5 \pmod 8$:
   (a) $C_0^{(-1)}=C_2$,
   (b) $C_2^{(-1)}=C_0$,
   (c) $C_3^{(-1)}=C_1$,
   (d) $C_1^{(-1)}=C_3$.
2. If $q\not\equiv 5 \pmod 8$ (hence $q\equiv 1 \pmod 8$):
   (a) $C_0^{(-1)}=C_0$,
   (b) $C_2^{(-1)}=C_2$,
   (c) $C_3^{(-1)}=C_3$,
   (d) $C_1^{(-1)}=C_1$.