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Lattices of lattice paths☆

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Abstract

We consider posets of lattice paths (endowed with a natural order) and begin the study of such structures. We give an algebraic condition to recognize which ones of these posets are lattices. Next we study the class of Dyck lattices (i.e., lattices of Dyck paths) and give a recursive construction for them. The last section is devoted to the presentation of a couple of open problems.

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1. Introduction

When a class of objects is introduced in mathematics, one of the first problems that naturally arises is to count how many objects there are. Typically, one recognizes an interesting numerical parameter and tries to enumerate the objects according to it, so obtaining a sequence of nonnegative integers. This is the main problem of enumerative combinatorics.

The second step (after mere enumeration) is to look for some "mathematical" structure (like, e.g., operations) the class of objects naturally possesses. As a matter of fact, one of the simplest structure that can be found is an order relation, as Gian-Carlo Rota suggested in his masterful paper (Rota, 1964). By the way, his theory of Möbius functions of posets leads

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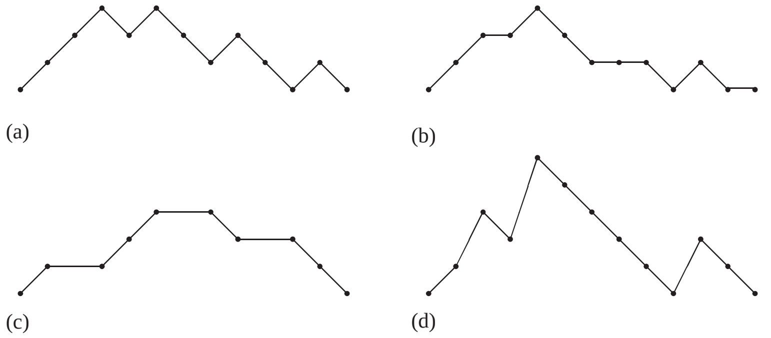

Fig. 1.

to many deep enumerative results, so proving that a better knowledge of the mathematical structure of a set gives a better insight even on his combinatorial and enumerative properties.

In this work we deal with classes of lattice paths (like, e.g., Dyck paths, Motzkin paths, Schröder paths, Lukasiewicz paths, etc.) which can be ordered in a completely natural manner. We postpone the formal definitions of these classes of paths at the beginning of Section 2. However, in Fig. 1 we give some instances of the paths which will more frequently occur in the present work. More precisely, we have drawn: (a) a Dyck path, (b) a Motzkin path, (c) a Schröder path and (d) a Lukasiewicz path.

Using a standard vocabulary, we say that Dyck paths only use rise (or $(1,1)$) steps and fall (or $(1,-1)$) steps, whereas Motzkin and Schröder paths also use horizontal steps of length 1 (or $(1,0)$ steps) and of length 2 (or $(2,0)$ steps), respectively. The case of Lukasiewicz paths is quite different, since they use an infinite set of possible steps, namely rise step of any type $(1,k)$ and (simple) fall steps of type $(1,-1)$.

Our goal is to begin a systematic study of the posets of paths arising in this way. A similar point of view has been undertaken in Autebert et al. (2002), where the authors study the posets arising from Delannoy paths; however, the present work deals with essentially different classes of paths. The only general problem we tackle here is to determine in which cases a poset of paths is a lattice, and we propose a possible solution to this problem. We also point out that the first works in this direction have been done by Narayana (see, for example, Narayana, 1979): in Narayana and Fulton (1958) a very interesting result is proved, described at the end of Section 2. Next, we focus on the study of a single type of paths, namely Dyck paths, and we provide an explicit (recursive) construction for the lattices of Dyck paths. It is then easy to see that a suitable, slight modification of such a construction can be successfully applied also to lattices of Schröder paths. We remark that in Latapy and Phan (to appear) and Latapy (2001) some similar problems are studied: the authors provide the construction of the lattice of partitions of a given integer n (with the dominance order) starting from the knowledge of the lattice of partitions of $n-1$. However, the methods used in the present work are completely different from those employed in Latapy and Phan (to appear) and Latapy (2001).

Quite surprisingly, our study sheds new light on a method of enumeration, usually called *ECO method* (ECO stands for “Enumeration of Combinatorial Objects”), which has proved