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Steady-state Markov chain models for certain q-confluent hypergeometric distributions

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Abstract

This paper draws attention to those members of the q-confluent hypergeometric family of discrete distributions that either (i) have special properties or (ii) arise as steady-state distributions from interesting Markov chains. They include (i) the Exton and O/U distributions and (ii) the q-hyper-Poisson I, Morse, confluent Bailey–Daum, and confluent q-Chu–Vandermonde distributions.

MSC: 60E99; 33D15; 60G50

Keywords: Heine distribution; q-hyper-Poisson I distribution; Morse distribution; Exton distribution; Third q-Poisson analogue; Confluent Bailey–Daum distribution; Confluent q-Chu–Vandermonde distribution; O/U distribution; Over-and-underdispersion; M/M/1 queues with balking; Birth–abort–death processes; Success-run processes; Current-age models; Success-run processes with stagnation

1. Introduction

The q-confluent family of discrete distributions has the probability generating function (pgf)

$$G_C(z) = {}_1\phi_1(b; c; q, -\lambda z)/{}_1\phi_1(b; c; q, -\lambda), \tag{1}$$

where $0 < q < 1$, $b < 1$, $c < 1$, and, for distributions with infinite support, $0 < \lambda$. Throughout this paper we adopt the Gasper and Rahman (1990) definition of a q-hypergeometric

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function as

$$
\begin{aligned}
&{}_A\phi_B(a_1,\ldots,a_A;b_1,\ldots,b_B;q,z)\\
&\quad=\sum_{j=0}^{\infty}\frac{(a_1;q)_j\ldots(a_A;q)_j z^j}{(b_1;q)_j\ldots(b_B;q)_j(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{B-A+1},
\end{aligned}
$$

where $(u;q)_j=(1-u)(1-uq)\cdots(1-uq^{j-1})$, $(u;q)_0=1$. When $|q|<1$ and $A=B$ the series converges for all z.

The q-confluent family is a generalization of the Heine distribution with pgf

$$
G_{\mathrm{H}}(z)={}_0\phi_0(-;-;q,-\lambda z)/{}_0\phi_0(-;-;q,-\lambda),\quad 0<q<1,\ 0<\lambda, \tag{2}
$$

of Benkherouf and Bather (1988). This arises as an infinite sum of independent Bernoulli rv's with log-linear odds. Kemp (1992a) explored its moments and other properties. Steady-state Markov chain models for it appeared in Kemp (1992b).

The purpose of the present paper is to draw attention to those special cases of (1) that are meaningful because either (i) they have special properties or (ii) they arise from interesting steady-state Markov chain models. Throughout we assume that $0<q<1$. Special cases that are extant in the literature are identified.

Section 2 looks at modifications of the stationary M/M/1 queue. The case $b=q$, $c=0$, $\lambda>0$, of (1) gives the distribution with probability mass function (pmf)

$$
p_x=\lambda^x q^{x(x-1)/2}\Bigg/\sum_{x=0}^{\infty}\lambda^x q^{x(x-1)/2}; \tag{3}
$$

setting $q=\gamma^2$ and $\lambda=\rho$ yields the Morse (1958) queue-size distribution for the M/M/1 queue with service-dependent balking. C.D. Kemp's (2002) q-hyper-Poisson-I distribution (q-HP-I) with pgf

$$
\begin{aligned}
&G_{q\mathrm{HPI}}(z)={}_1\phi_1(q;q^r;q,-q^{r-1}\theta z)/{}_1\phi_1(q;q^r;q,-q^{r-1}\theta),\\
&\quad 0<q<1,\ \ 0<r,\ \ 0<\theta,
\end{aligned} \tag{4}
$$

is the special case of (1) with $b=q$, $c=q^r$, and $\lambda=q^{r-1}\theta$, $0<\theta$. C.D. Kemp provided a group-size model for $zG_{q\mathrm{HPI}}(z)$. Section 2 models $G_{q\mathrm{HPI}}(z)$ using the M/M/1 queue with a modified form of service-dependent balking.

The well-known q-exponential functions,

$$
e_q(z)={}_1\phi_0(0;-;q,z)=1/(z;q)_\infty,\quad |z|<1,
$$

and

$$
E_q(z)={}_0\phi_0(-;-;q,-z)=(-z;q)_\infty,\quad |z|>0,
$$

lead to the Euler and Heine distributions, respectively. In Section 3 a third q-Poisson distribution is constructed via Exton's (1983) q-exponential function

$$
E(q,z)=\sum_{j=0}^{\infty}\frac{(1-q)^j z^j q^{j(j-1)/4}}{(q;q)_j},\quad E(q,z)=E(q^{-1},z).
$$