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Dual processes to solve single server systems

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Abstract

A new approach is used to determine the transient probability functions of the classical queueing systems: M/M/1, M/M/1/H, and M/M/1/H with catastrophes. This new solution method uses dual processes, randomization and lattice path combinatorics. The method reveals that the transient probability functions for M/M/1/H and M/M/1/H with catastrophes have the same mathematical form.

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1. Introduction

For over 50 years, determining new methods to obtain the transient probability functions of the classical single server queueing systems has captured and maintained the interest of theorists and practitioners alike. In this article, a new sample path approach that combines dual processes, randomization and lattice path combinatorics is used to obtain the transient probability functions of three single server queueing systems.

Section 2 contains background material and some important results connecting the transition probability functions of a birth–death process and its dual process. The transient

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probability functions of the classical M/M/1 queueing system are re-derived using dual processes and the reflection principle in Section 3. In Section 4, the transient probability functions of the M/M/1/H system are determined and formulated in terms of dual processes. The solution method again relies upon lattice path combinatorics instead of the traditional eigenvalue approach. Section 4 also contains an interesting formula for counting the number of lattice paths going from state j to state k in n steps confined within a given horizontal strip.

The dual process approach generalizes to certain non birth–death processes. In Section 5, the transient probability functions of an M/M/1/H system with catastrophes is determined. The analysis is surprisingly related to a suitably modified solution of the M/M/1/H system as described in Section 4. In this way, the dual process/randomization/lattice path combinatorics approach may unify the complicated analysis of finding transient probability functions for well-known Markovian queueing systems.

2. Dual processes

Consider a general birth–death process having transition birth rates λ_i for $i = 0, 1, 2, \ldots$ and transition death rates μ_i for $i = 1, 2, 3, \ldots$ as shown in the state rate transition diagram, Fig. 1 . All of these rates are assumed to be nonnegative numbers. The state space may be finite or countable.

Our interest is to determine $P_{i,j}(t)$, the transient probability functions where $i, j = 0, 1, 2, 3, \ldots$. For a finite or countable state space with uniformly bounded transition rates, $P_{i,j}(t)$ is determined by solving the Kolmogorov backward or forward equations, see Bhattacharya and Waymire (1990) or Gross and Harris (1985). The result is a system of differential equations that may be written in matrix form as

$$P'(t) = Q \cdot P(t) = P(t) \cdot Q,$$

where

$$P(t) = \begin{bmatrix} P_{0,0}(t) & P_{0,1}(t) & \cdot & \cdot & \cdot & P_{0,n}(t) & \cdot & \cdot & \cdot \\ P_{1,0}(t) & P_{1,1}(t) & \cdot & \cdot & \cdot & P_{1,n}(t) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & & & \\ \cdot & \cdot & & \cdot & & \cdot & & & \\ \cdot & \cdot & & & \cdot & \cdot & & & \cdot \\ P_{n,0}(t) & P_{n,1}(t) & & & & P_{n,n}(t) & & & \\ \cdot & \cdot & & & & \cdot & \cdot & & \\ \cdot & \cdot & & & & \cdot & & \cdot & \\ \cdot & \cdot & & & & \cdot & & & \cdot \end{bmatrix}$$

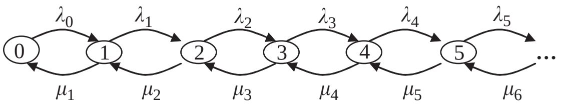

Fig. 1.