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Super-simple balanced incomplete block designs with block size 4 and index 6☆

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Abstract

Super-simple designs are useful in the construction of superimposed codes. The necessary condition for the existence of a super-simple balanced incomplete block design on v points with $k=4$ and $\lambda=6$, is that $v \geqslant 14$. The condition is shown to be sufficient.

MSC: 05B15; 62K10

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1. Introduction

A *group divisible design* (or GDD), is a triple $(X, \mathscr{G}, \mathscr{B})$ which satisfies the following properties:

1. $\mathscr{G}$ is a partition of a set X (of *points*) into subsets called *groups*;
2. $\mathscr{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point;
3. Every pair of points from distinct groups occurs in exactly λ blocks.

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The *group type* (or *type*) of GDD is the multiset $\{|G| : G \in \mathscr{G}\}$. We shall use an "exponential" notation to describe types: so type $g_1^{u_1} \cdots g_k^{u_k}$ denotes u_i occurrences of g_i, $1 \leqslant i \leqslant k$, in the multiset. A GDD with block sizes from a set of positive integers K is called a (K, λ)-GDD. When $K = \{k\}$, we simply write k for K. When $\lambda = 1$, we simply write K-GDD. A (k, λ)-GDD with group type 1^v is called a *balanced incomplete block design*, denoted by (v, k, λ)-BIBD.

A design is called *simple* if it contains no repeated blocks. A design is said to be *super-simple* if the intersection of any two blocks has at most two elements. When $k = 3$, a super-simple design is just a simple design. When $\lambda = 1$, the designs are necessarily super-simple. In this paper, when we talk about super-simple BIBDs, we usually mean the case $k \geqslant 4$ and $\lambda > 1$.

The term super-simple designs was introduced by Gronau and Mullin (1992). The existence of super-simple designs is an interesting extremal problem by itself, but there are also some useful application. Such designs are used in coverings (Bluskov and Hämäläinen, 1998), in the construction of new designs (Bluskov, 1997) and in the construction of superimposed codes (Kim and Lebedev, 2004).

It is well known that the following are the necessary conditions for the existence of a super-simple (v, k, λ)-BIBD:

1. $v \geqslant (k-2)\lambda + 2$;
2. $\lambda(v-1) \equiv 0 \pmod{k-1}$;
3. $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

For the existence of super-simple $(v, 4, \lambda)$-BIBDs, the necessary conditions are known to be sufficient for $\lambda = 2, 3, 4$. Gronau and Mullin solved the case for $\lambda = 2$ (Gronau and Mullin, 1992), and the corrected proof appeared in Khodkar (1994). The $\lambda = 3$ case was solved independently by Khodkar (1994) and Chen (1995). The $\lambda = 4$ case was solved independently by Adams et al. (1996) and Chen (1996). A recent survey on super-simple $(v, 4, \lambda)$-BIBDs with $v \leqslant 32$ can be found in Bluskov and Heinrich (2001). We summarize these known results in the following theorem.

Theorem 1.1 (*Gronau and Mullin, 1992; Khodkar, 1994; Chen, 1995, 1996; Adams et al., 1996*). *A super-simple $(v, 4, \lambda)$-BIBD exists for $\lambda = 2, 3, 4$ if and only if the following conditions are satisfied*:

1. $\lambda = 2$, $v \equiv 1 \pmod 3$ *and* $v \geqslant 7$;
2. $\lambda = 3$, $v \equiv 0, 1 \pmod 4$ *and* $v \geqslant 8$;
3. $\lambda = 4$, $v \equiv 1 \pmod 3$ *and* $v \geqslant 10$.

In this paper we investigate the existence of super-simple $(v, 4, 6)$-BIBDs. Clearly, when $k = 4$ and $\lambda = 6$ the necessary condition becomes $v \geqslant 14$. We shall use direct and recursive constructions to show that this necessary condition is also sufficient. Specifically, we shall prove the following.

Theorem 1.2. *A super-simple $(v, 4, 6)$-BIBD exists if and only if $v \geqslant 14$.*