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Journal of Statistical Planning and Inference 128 (2005) 311–325

journal of statistical planning and inference

www.elsevier.com/locate/jspi

On orthogonal designs in order 48

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Received 29 August 2002; accepted 29 October 2003

Abstract

We show that all 3164 possible OD($48; s_1, s_2, s_3$) exist. In addition to the use of some classical techniques we employ two new methods of construction.

MSC: primary 05B20; secondary 62K05; 62K10

Keywords: Weighing matrices; Orthogonal designs; Autocorrelation; Amicable set of matrices; Additive property for matrices

1. Introduction

An *orthogonal design A*, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted OD($n; s_1, s_2, \ldots, s_u$) on the commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_u$ is a square matrix of order n with entries $\pm x_k$ or 0, where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words

$$AA^{\mathrm{T}} = (s_1 x_1^2 + \cdots + s_u x_u^2) I_n,$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d$, $0 \leqslant d < 4$, then $\rho(n) = 8c + 2^d$.

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[1] Supported by an NSERC grant.

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2003.10.001

If A and B are orthogonal designs of the same order n and of types $(s_1, s_2, \ldots, s_u)$ and $(t_1, t_2, \ldots, t_v)$, respectively, and if $AB^{\mathrm{T}} = A^{\mathrm{T}}B$ then A and B are called *amicable orthogonal designs*, denoted $\mathrm{AOD}(n; (s_1, s_2, \ldots, s_u); (t_1, t_2, \ldots, t_v))$. If in addition, C is an orthogonal design of order n and type $(p_1, p_2, \ldots, p_w)$, if $A * C = B * C = 0$, where $*$ denotes the Hadamard product, and if $A + C$ and $B + C$ are orthogonal designs, then A, B, C is called a *product design* of order n and types $(s_1, s_2, \ldots, s_u)$, $(t_1, t_2, \ldots, t_v)$, $(p_1, p_2, \ldots, p_w)$, denoted $\mathrm{PD}(n; (s_1, s_2, \ldots, s_u; t_1, t_2, \ldots, t_v; p_1, p_2, \ldots, p_w))$.

It is conjectured that all possible 3-tuples are types of an orthogonal design of order $8n$. This conjecture has been verified for $n = 1, 2, 3, 4$. In this paper we will show that the conjecture also holds for $n = 6$. The case $n = 5$ does not seem to be as tractable. In addition to the use of some classical techniques we employ two new methods of construction.

Section 2 is devoted to classical results. By classical results we mean all orthogonal designs obtained more than 25 years ago. Out of 3164 possible cases 3147 are constructed using classical results. To show the evolution of the subject area, we keep the new results out of this section. By new results we mean results obtained in the past 4 years.

2. Classical results

For this part we make extensive use of the book of Geramita and Seberry (1979). For convenience we quote a number of construction methods from this book.

Theorem 1 (Equating and Killing Theorem). *If A is an orthogonal design $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_u$, then there is an orthogonal design $\mathrm{OD}(n; s_1, s_2, \ldots, s_i + s_j, \ldots, s_u)$ and $\mathrm{OD}(n; s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_u)$ on the $u-1$ commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_{j-1}, \pm x_{j+1}, \ldots, \pm x_u$.*

Theorem 2 (Multiplication Theorem). *If there exists an orthogonal design $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$, then there exists an orthogonal design $\mathrm{OD}(2n; s_1, s_1, es_2, \ldots, es_u)$, where $e=1$ or 2.*

Theorem 3 (Second Multiplication Theorem). *If there exists an orthogonal design $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$, then there exists an orthogonal design $\mathrm{OD}(2n; e_1s_1, e_2s_2, \ldots, e_us_u)$, where $e_i = 1$ or 2, $i = 1, \ldots, u$.*

Theorem 4. *We use the following designs in order 24 (Table 1) and the Multiplication Theorems above, to obtain the designs given in the appendix which have 9, 8, and 7 variables in order 48.*

Theorem 5. *An $\mathrm{OD}(6; a, b)$ and an $\mathrm{AOD}(8; (a_1, a_2, \ldots, a_s); (b_1, b_2, \ldots, b_t))$ give an $\mathrm{OD}(48; aa_1, aa_2, \ldots, aa_s, bb_1, bb_2, \ldots, bb_t)$.*

Table 2 gives the designs in order 48 constructed using Theorem 5.