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Asymptotic behavior of the maximum from distributions subject to trends in location and scale

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ABSTRACT

Suppose we observe $Y_{jn} = \mu_{jn} + \sigma_{jn} e_j$ for $1 \le j \le n$ in $\mathbb{R}$, where $\{e_j\}$ are independent and identical random errors with common distribution function $F(x)$. Let $M_n = \max_{1 \le j \le n} Y_{jn}$. When the upper tail of F is of power-type, local power type, gamma type and normal type, we give conditions on the growth of the location and scale trends $\{\mu_{jn}, \sigma_{jn}\}$ such that for certain constants a_n and $b_n > 0$, $b_n M_n - a_n$ converges to one of the three standard extreme value distributions. In each case b_n is proportional to the L_p-norm of $\{\sigma_{jn}\}$ and does not depend on $\{\mu_{jn}\}$. Most importantly, trend in scale is shown to dominate trend in location.

1. Introduction and summary

We characterize the asymptotic behavior of the maximum of a random sample subject to trends in both location and scale. In each case considered the limit is one of the three classical extreme value (EV) distributions obtained by Fisher and Tippett (1927) for the case of no trends. Suppose we observe $Y_{jn} = \mu_{jn} + \sigma_{jn} e_j$ for $1 \le j \le n$ in $\mathbb{R} = (-\infty, \infty)$, where $\{e_j\}$ are independent and identical random errors with common distribution function $F(x)$ on $\mathbb{R}$, and $\{\mu_{jn}, \sigma_{jn}\}$ are constants giving the trends in location and scale. Such models are useful, for example, for long term temperature series, where both the mean annual temperature and the variability are changing. We shall show that the trend in scale is much more important in determining the behavior of the sample maximum, $M_n = \max_{1 \le j \le n} Y_{jn}$, than the trend in location. For the most important types of upper tail behavior for $F(x)$, and a wide class of $\{\mu_{jn}, \sigma_{jn}\}$, we shall give explicit standardizing constants a_n and $b_n > 0$ such that

$$M_n' = b_n M_n - a_n \xrightarrow{\mathcal{L}} Y \tag{1.1}$$

as $n \to \infty$, where Y has one of the three EV distributions:

$$G_0(x) = P(Y_0 < x) = \exp\{-\exp(-x)\} \quad \text{on } \mathbb{R}, \ \text{(Gumbel)}, \tag{1.2}$$

$$G_\theta(x) = P(Y_\theta < x) = \exp\left\{-x^{-\theta}\right\} \quad \text{on } (0, \infty), \ \text{(Fréchet)}, \tag{1.3}$$

$$\overline{G}_\theta(x) = P\left(\overline{Y}_\theta < x\right) = \exp\left\{-(-x)^\theta\right\} \quad \text{on } (-\infty, 0), \ \text{(Weibull)}. \tag{1.4}$$

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These are also known as the EV1, EV2 and EV3 distributions. Note that $\theta > 0$ in (1.3) and (1.4). In each case b_n^{-1} is a function of scale alone and can be taken as a multiple of

$$\|\sigma_n\|_p = \left(n^{-1}\sum_{j=1}^{n}\sigma_{jn}^p\right)^{1/p} \quad \text{or} \quad \|\sigma_n\|_\infty = \max_{1\le j\le n}\sigma_{jn}, \tag{1.5}$$

where $p = \infty$ for EV1, $p = \theta$ for EV2, and $p = -\theta$ for EV3. In Section 2, we show convergence to EV2 for a *power tail:*

$$1 - F(x) = Kx^{-\theta}\{1 + o(1)\} \tag{1.6}$$

as $x \to \infty$, where $\theta > 0$ with $a_n = 0$. In Section 3, we show convergence to EV3 for a *local power tail:*

$$1 - F(x) = K(-x)^{\theta}\{1 + o(1)\} \tag{1.7}$$

as $x \uparrow 0$, where $\theta > 0$ with $a_n = 0$. In Section 4, we show convergence to EV1 for a *gamma tail:*

$$1 - F(x) = Kx^{-\theta}\exp(-x)\{1 + o(1)\} \tag{1.8}$$

as $x \to \infty$, and for a *normal tail:*

$$1 - F(x) = Kx^{-\theta}\exp\left(-x^2\right)\{1 + o(1)\} \tag{1.9}$$

as $x \to \infty$. The results for a gamma tail are more complex: they depend on how σ_{jn} behaves when near its maximum. We need to distinguish between the "continuous" case when σ_{jn} may be arbitrarily close to

$$\sigma_{\infty n} = \max_{1\le j\le n}\sigma_{jn} \tag{1.10}$$

for sufficiently large n and the "discontinuous" case when $\lambda_n = \sigma_{\infty n}/\max_{1\le j\le n}\{\sigma_{jn} : \sigma_{jn} < \sigma_{\infty n}\}$ is bounded away from one. When studying the continuous case we shall assume that

$$\mu_{jn} = \mu(j/n), \qquad \sigma_{jn} = \sigma(j/n) \tag{1.11}$$

for some functions $\mu(t), \sigma(t)$ on [0, 1]. We then need to consider the behavior of $\sigma(t)$ in a neighborhood of $\{t_0\}$ maximizing $\sigma(t)$. Of the many possibilities we confine ourselves to the following:

(i) $\sigma(t)$ is constant;
(ii) $\sigma(t)$ has a unique maximum at t_0, where $0 < t_0 < 1$ and $\sigma^{(1)}(t_0) = 0 > \sigma^{(2)}(t_0)$, where $\sigma^{(i)}(t)$ is the ith derivative of $\sigma(t)$;
(iii) $\sigma(t)$ has a unique maximum at $t_0 = 0$ or 1, and $\sigma^{(1)}(t_0) \neq 0$.

For a normal tail there are yet more possibilities in the continuous case (1.11) if $\{t_0\}$ includes an interval, $t_1 \le t_0 \le t_2$, say (for example, if $\sigma(t) \equiv 1$): we then need to consider how $\mu(t)$ behaves near $\{t_0^*\}$ maximizing $\mu(t)$ in $[t_1, t_2]$; for brevity we confine ourselves to the following: (ii) and (iii) above,

(iv) $\sigma(t) \equiv 1$, $\mu(t)$ has a unique maximum at t_0^*, where $0 < t_0^* < 1$ and $\mu^{(1)}(t_0^*) = 0 > \mu^{(2)}(t_0^*)$;
(v) $\sigma(t) \equiv 1$, $\mu(t)$ has a unique maximum at $t_0^* = 0$ or 1, and $\mu^{(1)}(t_0^*) \neq 0$.

For more general cases, where $\{t_0\}$ or $\{t_0^*\}$ is a finite set or when say in (ii), $\sigma^{(i)}(t_0) = 0$ for $i < 2m$ and $\sigma^{(2m)}(t_0) < 0$, one may adapt these results by applying the saddlepoint results of Withers and Nadarajah (2013b).

Returning to the discontinuous case, we shall assume that for some $\lambda > 1$ not depending on n

$$\sigma_{\infty n}/\max_{1\le j\le n}\left(\sigma_{jn} : \sigma_{jn} < \sigma_{\infty n}\right) > \lambda$$

for $\sigma_{\infty n}$ of (1.10). Examples of this type include $\sigma_{jn} \equiv \sigma_j$ periodic and $\{\sigma_{jn} \equiv \sigma_j$, where $\sigma_j/\sigma_{j-1} > \lambda$ for $j \ge 1\}$. The behavior of M_n then depends on $p_n = \#\{j : 1 \le j \le n, \sigma_{jn} = \sigma_{\infty n}\}$, the number of maximizers σ_{jn} has. We shall require that $p_n \to \infty$ sufficiently fast. There is a tradeoff between how fast $\{\mu_{jn}\}$ may grow and how slowly the number of maximizers p_n may grow: the weakest conditions on $\{\mu_{jn}\}$, $\max_{1\le j\le n}|\mu_{jn}| = O(\log n)$, occur when $p_n/n \to 1$; see Theorem 4.2. This confirms a conjecture in Withers (1995) for the case when $\sigma_{jn} \equiv \sigma_j$ is periodic, and so provides a connection to results of Ballerini and McCormick (1989) who considered this case. Our results for power and gamma tails extend those given for the continuous case in Withers (1995). Extensions to expansions may be obtained in principle by assuming an expansion for the upper tail of F, as done in Withers and Nadarajah (2013a,c) for power and gamma tails with $\sigma_{jn} \equiv 1$, $\mu_{jn} = \mu(j/n)$. In particular by replacing $o(1)$ in the tail expressions (1.6)–(1.9) above by a magnitude like $O(x^{-\beta})$, we shall obtain an explicit δ_n such that $M_n' - Y = O_p(\delta_n)$ in the sense that

$$P\left(M_n' \le x\right) - P(Y \le x) = O(\delta_n) \tag{1.12}$$

for all x.

These results can be extended to multivariate observations. They should also be extendable to correlated residuals $\{e_j, 1 \le j \le n\}$ and to random processes in continuous time $Y(t) = \mu(t) + \sigma(t)e(t)$, $0 \le t \le T$, where $e(t)$ is say a