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The total variation distance between two double Wiener–Itô integrals

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ABSTRACT

Using an approach recently developed by Nourdin and Poly (2013), we improve the rate in an inequality for the total variation distance between two double Wiener–Itô integrals originally due to Davydov and Martynova (1987). An application to the rate of convergence of a functional of a correlated two-dimensional fractional Brownian motion towards the Rosenblatt random variable is then given, following a previous study by Maejima and Tudor (2012).

1. Introduction

Suppose that $X = \{X(h),\ h \in \mathfrak{H}\}$ is an isonormal Gaussian process on a real separable infinite-dimensional Hilbert space $\mathfrak{H}$. For any integer $p \geqslant 1$, let $\mathfrak{H}^{\otimes p}$ be the pth tensor product of $\mathfrak{H}$. Also, denote by $\mathfrak{H}^{\odot p}$ the pth symmetric tensor product.

The following statement is due to Davydov and Martynova (1987); see also Nourdin and Poly (2013, Theorem 4.4).

Theorem 1.1. *Fix an integer $p \geqslant 2$, and let (f_n) be a sequence of $\mathfrak{H}^{\odot p}$ that converges to f_∞ in $\mathfrak{H}^{\otimes p}$. Assume moreover that f_∞ is not identically zero. Let $I_p(f_n)$, $n \in \mathbb{N} \cup \{\infty\}$, denote the pth Wiener–Itô integral of f_n with respect to X. Then, there exists $c > 0$ such that, for all n,*

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_p(f_n) \in C) - P(I_p(f_\infty) \in C)| \leqslant c\, \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes p}}^{1/p}, \tag{1.1}$$

where $\mathcal{B}(\mathbb{R})$ stands for the set of Borelian sets of $\mathbb{R}$.

In this work, $p = 2$ and the inequality (1.1) becomes

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| \leqslant c\sqrt{\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}}}. \tag{1.2}$$

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With each $f_\infty \in \mathfrak{H}^{\odot 2}$, one may associate the following Hilbert–Schmidt operator:

$$A_{f_\infty} : \mathfrak{H} \to \mathfrak{H}, \quad g \mapsto \langle f_\infty, g \rangle_{\mathfrak{H}}. \tag{1.3}$$

Let $\lambda_{\infty,k}$, $k \geqslant 1$, indicate the eigenvalues of A_{f_∞}. In many situations of interest (see below for an explicit example), it happens that the following property, that we label for further use, is satisfied for f_∞:

$$\text{the cardinality of } \{k : \lambda_{\infty,k} \neq 0\} \text{ is at least } 5. \tag{1.4}$$

The aim of this work is to take advantage of (1.4) in order to improve (1.2) by a factor 2. More precisely, relying on an approach recently developed by Nourdin and Poly (2013), we shall prove the following result—compare with (1.2):

Theorem 1.2. *Let f_∞ be an element of $\mathfrak{H}^{\odot 2}$ satisfying (1.4) (in particular, f_∞ is not identically zero). Let (f_n) be a sequence of $\mathfrak{H}^{\odot 2}$ that converges to f_∞ in $\mathfrak{H}^{\otimes 2}$. Then, there exists $c > 0$ (depending only on f_∞) such that, for all n,*

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| \leqslant c \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}}. \tag{1.5}$$

In some sense, the inequality (1.5) appears to be optimal. Indeed, consider $F_\infty = I_2(f_\infty)$ with f_∞ satisfying (1.4) and set $F_n = I_2(f_n)$ with $f_n = (1 + c_n) f_\infty$, where (c_n) is a sequence of nonzero real numbers converging to zero. Let ϕ_∞ (resp. ϕ_n) denote the density of F_∞ (resp. F_n), which exists thanks to Shigekawa's theorem (see Shigekawa, 1980). Assume furthermore that ϕ_∞ is differentiable and is such that $0 < \int_{\mathbb{R}} |x\phi'_\infty(x) + \phi_\infty(x)| dx < \infty$. According to Scheffé's theorem, one has

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| = \frac{1}{2} \int_{\mathbb{R}} |\phi_n(x) - \phi_\infty(x)| dx.$$

We deduce, after some easy calculations, that

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| \sim_{n\to\infty} \frac{1}{2} |c_n| \int_{\mathbb{R}} |x\phi'_\infty(x) + \phi_\infty(x)| dx.$$

On the other hand, $\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}} = |c_n| \, \|f_\infty\|_{\mathfrak{H}^{\otimes 2}}$. Thus,

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| \sim_{n\to\infty} c \, \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}},$$

with $c = \int_{\mathbb{R}} |x\phi'_\infty(x) + \phi_\infty(x)| dx / (2\|f_\infty\|_{\mathfrak{H}^{\otimes 2}})$.

To illustrate the use of Theorem 1.2 in a concrete situation, we consider the following example taken from Maejima and Tudor (2012). Let B^{H_1}, B^{H_2} be two fractional Brownian motions with Hurst parameters $H_1, H_2 \in (0, 1)$, respectively. We assume that both H_1 and H_2 are strictly bigger than $\frac{1}{2}$. We further assume that the two fractional Brownian motions B^{H_1} and B^{H_2} can be expressed as Wiener integrals with respect to the *same* two-sided Brownian motion W, meaning in particular that B^{H_1} and B^{H_2} are *not* independent. To be precise, we set

$$B_t^{H_1} = c(H_1) \int_{\mathbb{R}} dW_y \int_0^t (u - y)_+^{H_1 - \frac{3}{2}} du, \quad t \geqslant 0, \tag{1.6}$$

$$B_t^{H_2} = c(H_2) \int_{\mathbb{R}} dW_y \int_0^t (u - y)_+^{H_2 - \frac{3}{2}} du, \quad t \geqslant 0, \tag{1.7}$$

where the constants $c(H_1)$ and $c(H_2)$ are chosen such that $E[(B_1^{H_1})^2] = E[(B_1^{H_2})^2] = 1$. Define

$$Z_n = n^{1-H_1-H_2} \sum_{k=0}^{n-1} \left[\frac{\left(B^{H_1}_{\frac{k+1}{n}} - B^{H_1}_{\frac{k}{n}}\right)\left(B^{H_2}_{\frac{k+1}{n}} - B^{H_2}_{\frac{k}{n}}\right)}{E\left[\left(B^{H_1}_{\frac{k+1}{n}} - B^{H_1}_{\frac{k}{n}}\right)\left(B^{H_2}_{\frac{k+1}{n}} - B^{H_2}_{\frac{k}{n}}\right)\right]} - 1 \right]. \tag{1.8}$$

When $H_1 = H_2 = H$, observe that (1.8) is related to the quadratic variation of B^H. In Maejima and Tudor (2012), the following extension of a classical result by Taqqu (1975) is shown:

Proposition 1.3. *Assume that $H_1 > \frac{1}{2}$, $H_2 > \frac{1}{2}$ and $H_1 + H_2 > \frac{3}{2}$. Then, Z_n converges as $n \to \infty$ in $L^2(\Omega)$ to the non-symmetric Rosenblatt random variable Z_∞, given by*

$$Z_\infty = b(H_1, H_2) \int_{\mathbb{R}^2} dW_x dW_y \int_0^1 (s - x)_+^{H_1 - 3/2} (s - y)_+^{H_2 - 3/2} ds. \tag{1.9}$$

Here $b(H_1, H_2)$ is a normalizing explicit constant whose precise value does not matter in the sequel.