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Conditionally negative association resulting from multinomial distribution[☆]

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ABSTRACT

Investigation on conditionally dependent random variables has undergone a considerable development, but not nearly as much treating practical applications. This note provides a concrete example of conditionally negative associated random variables resulting from multinomial distribution and simultaneously confirms a conjecture in the literature.

1. Introduction

We will be working on a fixed probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. Random events $A_1, A_2, \ldots, A_n$ are said to be conditionally independent given $\mathcal{F}$ ($\mathcal{F}$-independent, in short) if

$$P\left(\cap_{i=1}^n A_i|\mathcal{F}\right) = \prod_{i=1}^n P\left(A_i|\mathcal{F}\right) \quad \text{a.s.},$$

while random variables $X_1, X_2, \ldots, X_n$ are said to be $\mathcal{F}$-independent if the classes of events $\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n)$ are $\mathcal{F}$-independent.

Of course, $\mathcal{F}$-independence reduces to the usual (unconditional) independence when $\mathcal{F} = \{\varnothing, \Omega\}$, but the independence of random events/variables does not imply $\mathcal{F}$-independence and the $\mathcal{F}$-independence of random events/variables also does not imply their independence for any sub-σ-algebra $\mathcal{F}$ of $\mathcal{A}$.

The statistical perspective of conditional independence is that of a Bayesian. A problem begins with a parameter Θ with its prior probability distribution that exists only in mind of the investigator. The statistical model that is most commonly in use is that of observable random variables $X_1, X_2, \ldots, X_n$ that are independently and identically distributed for each given value of Θ. In addition, conditional independence has been fruitfully applied in some important concepts of statistics,

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for example, sufficiency and ancillarity can be regarded as expressions of conditional independence, and many results and theorems concerning these concepts are just applications of some simple general properties of conditional independence.

Following Prakasa Rao (2009) for the sake of convenience we will use the notation $P^{\mathcal{F}}(A)$ to denote $P(A|\mathcal{F})$ and $E^{\mathcal{F}}X$ to denote $E(X|\mathcal{F})$. In addition, $\mathrm{Cov}^{\mathcal{F}}(X, Y)$ denotes the conditional covariance of X and Y given $\mathcal{F}$, i.e.,

$$\mathrm{Cov}^{\mathcal{F}}(X, Y) = E^{\mathcal{F}}(XY) - E^{\mathcal{F}}X \cdot E^{\mathcal{F}}Y.$$

One of many possible generalizations from conditional independence to conditional dependence is conditionally negative association introduced by Roussas (2008), a more in-depth discussion on this topic can be found in Yuan et al. (2010).

Random variables $X_1, X_2, \ldots, X_n$ are said to be conditionally negatively associated given $\mathcal{F}$ ($\mathcal{F}$-NA, in short) if, for every pair of disjoint subsets I and J of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}^{\mathcal{F}}\left(f\left(X_i, i \in I\right), g\left(X_j, j \in J\right)\right) \leq 0 \quad \text{a.s.}$$

whenever f and g are coordinatewise nondecreasing and the $\mathcal{F}$-covariance exists (it is convenient to allow the case $I = \varnothing$ setting $f(X_i, i \in I) = 0$). A sequence $\{X_n, n \geq 1\}$ of random variables is said to be $\mathcal{F}$-NA if every finite subsequence is $\mathcal{F}$-NA.

Obviously, conditionally negative association turns into negative association (NA) if the conditional σ-algebra is taken as $\{\varnothing, \Omega\}$, but the relation between conditionally negative association and negative association is just like the relation between conditional independence and independence referred to earlier. More specifically, conditionally negative association of random variables does not imply negative association and the opposite implication is also not true, such examples are established in Yuan et al. (2010).

Majerek et al. (2005) established a conditional version of the Kolmogorov strong law of large numbers under the framework of a conditional σ-algebra. The subject of conditional independence/dependence has since then undergone a considerable development, not to say boom. See, for instance, Christofides and Hadjikyriakou (2013) for conditional demimartingale, Liu and Prakasa Rao (2013) for conditional Borel–Cantelli lemma, Ordóñez Cabrera et al. (2012) for conditionally negative quadrant dependence, Prakasa Rao (2009) and Yuan and Yang (2011) for conditional association, Wang and Wang (2013) for conditional demimartingale and conditional N-demimartingale, Yuan et al. (2010) for conditionally negative association, Yuan and Lei (2013) for conditionally strong mixing, Yuan and Xie (2012) for conditionally linearly negative quadrant dependence, and so forth.

Conditionally negative association is considered as important part of conditional dependence, and many elegant results are available, but not nearly as much treating practical applications. What is done in this note is to give an example of conditionally negative association.

Suppose that we perform a sequence of n independent and identical experiments and each experiment can result in any one of m ($m \geq 3$) possible outcomes, with respective probabilities $p_1, p_2, \ldots, p_m$ (of course these probabilities are subject to the condition $\sum_{i=1}^{m} p_i = 1$).

For $i = 1, 2, \ldots, m$, let X_i be the random variable denoting the number of the n experiments that result in outcome number i, then random vector $(X_1, X_2, \ldots, X_m)$ has the well-known multinomial distribution with parameters n and $(p_1, p_2, \ldots, p_m)$, its joint probability mass function is given by

$$P\left(X_1 = n_1, X_2 = n_2, \ldots, X_m = n_m\right) = \frac{n!}{n_1! n_2! \cdots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

whenever $\sum_{i=1}^{m} n_i = n$.

Joag-Dev and Proschan (1983) proved that $X_1, X_2, \ldots, X_m$ are NA. Furthermore, Yuan et al. (2010) made the following

Conjecture 1.1. *Let $2 \leq r < m$ be given. Then $X_1, X_2, \ldots, X_r$ are $\sigma(X_{r+1}, \ldots, X_m)$-NA.*

2. Intuitive explanations of the conjecture

Although the n experiments are independent and identical, the random variables $X_1, X_2, \ldots, X_m$ are not independent; in particular, their sum is fixed to n and therefore $\sum_{i=1}^{r} X_i = n - \sum_{i=r+1}^{m} X_i$.

A fundamental natural question of interest is: how are these X_i related? Let $X_{r+1}, \ldots, X_m$ be given. Intuitively, if one part of $X_1, X_2, \ldots, X_r$ is "high", then the other part is "low", namely $X_1, X_2, \ldots, X_r$ are $\sigma(X_{r+1}, \ldots, X_m)$-NA. However, establishing such an assertion precisely by a direct calculation from the joint probability mass function, though possible in principle, appears to be quite a formidable task.

3. Proof of the conjecture

The following result is a analogue of yielding result by Kimball (1951) under the conditioning setup.

Lemma 3.1. *Let $u(x)$ and $v(x)$ be both nondecreasing functions. Then*

$$\mathrm{Cov}^{\mathcal{F}}\left(u(X), v(X)\right) \geq 0 \quad \textit{a.s.}$$

with X being any random variable and $\mathcal{F}$ being any sub-σ-algebra of $\mathcal{A}$, whenever the conditional covariance exists.