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Tail conditional expectation for multivariate distributions: A game theory approach

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ABSTRACT

This paper proposes using the Shapley values in allocating the total tail conditional expectation (TCE) to each business line (X_j, $j = 1, \dots, n$) when there are n correlated business lines. The joint distributions of X_j and S ($S = X_1 + X_2 + \cdots + X_n$) are needed in the existing methods, but they are not required in the proposed method.

1. Introduction

A risk measure is defined as a mapping from the set of random variables representing the risk exposure to a real number. The well-known risk measures in the literature are value at risk (VaR), tail conditional expectation (TCE), and shortfall expectation (SE). Let X denote the possible loss of a portfolio at a given time horizon. Then $\mathrm{VaR}_X(1-\alpha)$ is the size of loss for which there is a small probability α for exceeding that loss (also shown by $x_{1-\alpha}$ or $\xi_{1-\alpha}$); therefore, $\mathrm{VaR}_X(1-\alpha)$ is defined as the smallest value x satisfying $\Pr(X > x) = \alpha$. The mathematical form of the value at risk, $\mathrm{VaR}_X(1-\alpha)$, is given by

$$\mathrm{VaR}_X(1-\alpha) = \inf\{x|\Pr(X > x) \leqslant \alpha\}. \tag{1.1}$$

The tail conditional expectation, $\mathrm{TCE}_X(1-\alpha)$, is the mean of worse losses, given that the loss will exceed a particular value $x_{1-\alpha}$. It is expressed by

$$\mathrm{TCE}_X(1-\alpha) = E[X|X > \mathrm{VaR}_X(1-\alpha)] = E(X|X > x_{1-\alpha}). \tag{1.2}$$

Finally, the shortfall expectation, $\mathrm{SE}_X(1-\alpha)$, is defined as

$$\mathrm{SE}_X(1-\alpha) = \mathrm{TCE}_X(1-\alpha) + x_{1-\alpha}(1-\alpha-\Pr(x \le x_{1-\alpha})). \tag{1.3}$$

$1-\alpha$ is called the confidence level, and in practice it is often set to 0.95 or 0.99. It follows from the definitions that $\mathrm{SE}_X(1-\alpha) \ge \mathrm{TCE}_X(1-\alpha) \ge \mathrm{VaR}_X(1-\alpha)$. When X is a continuous random variable, then $\Pr(X \le \mathrm{VaR}_X(1-\alpha)) = 1-\alpha$ and $\mathrm{SE}_X(1-\alpha)$ is equal to $\mathrm{TCE}_X(1-\alpha)$. When compared to the VaR measure, the TCE provides a more conservative measure of risk for the same degree of confidence level, and it provides an effective tool for analyzing the tail of the loss distribution. In multivariate cases, assume that a company manages n lines of business and that the risk managers of that company estimate the aggregated risk of all business lines and are interested to know how much risk is concealed

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in each business line. Let X_j denote the jth loss variable ($j = 1, \ldots, n$). If $\zeta_S(1-\alpha)$ indicates the risk measure for S, where $S = X_1 + X_2 + \cdots + X_n$, we would like to determine $\zeta'_{X_j}(1-\alpha)$ as the risk measure for X_j such that

$$\zeta_S(1-\alpha) = \sum_{i=1}^{n} \zeta'_{X_j}(1-\alpha). \tag{1.4}$$

In recent years, attention has turned to coherent risk measurements. A risk measure (ζ) is called a coherent risk measure if, and only if, it satisfies all of the following four axioms (Artzner et al., 1999).

- Sub-additivity: This means that the risk of two, or more, portfolios together cannot get any worse than adding the two, or more, risks separately; this is the diversification principle.
- Positive homogeneity: $\zeta(\lambda X) = \lambda \zeta(X)$ for $\lambda \geq 0$.
- Translation invariance: $\zeta(X + a) = \zeta(X) + a$ for any $a \in \mathbb{R}$.
- Monotonicity: If $X_1 \leq X_2$, then $\zeta(X_1) \leq \zeta(X_2)$. This means that if portfolio X_2 always has better values than portfolio X_1 under all scenarios, then the risk of X_2 should be less than the risk of X_1.

It is well known that the VaR fails to satisfy the coherency principle. In general, the VaR is not a coherent risk measure, as it violates the sub-additivity principle and often underestimates the tail risk. An immediate consequence is that the VaR might discourage diversification (Artzner et al., 1999). Zhu and Li (2012) studied the asymptotic relation between the TCE and the VaR and showed that, for a large class of continuous heavy-tailed risks, the TCE is asymptotically proportional to the VaR of aggregation, given that the aggregated risk exceeds a large threshold. It is proven that the SE is a coherent risk measure; therefore, the TCE is a coherent measure for continuous distributions.

In this study, we consider the TCE since it exhibits properties that are considered desirable and applicable in a variety of situations. To find the risk concealed in each individual variable in multivariate environments, we use the cooperative game theory concept, and apply the Shapley value decomposition to calculate the TCE for each variable. In existing approaches to estimate the risk share for each variable from the total risk, the joint distribution of X_j ($j = 1, \ldots, n$) and sum of all variables (S) is required, where estimating the joint distribution is not a straightforward task. The proposed method uses Shapley values in a cooperative game theory approach to allocate the total TCE fairly to its constituents without the need to fit any joint distributions.

The remainder of the paper is organized as follows. Section 2 presents the existing approaches in estimating risk measures in multivariate environments. Section 3 reviews the concepts of the cooperative game theory and Shapley values. Section 4 discusses the concept of Shapley values in risk allocation and describes the proposed method. Several numerical examples for multivariate normal and non-normal distributions are illustrated in Section 5. Finally, Section 6 concludes.

2. TCE for multivariate distributions

In multivariate cases, where we have multiple lines of correlated business ($X_j, j = 1, \ldots, n$), the total TCE is calculated from

$$\text{TCE}_S(1-\alpha) = E\left(S = \sum_{i=1}^{n} X_i | S > s_{1-\alpha}\right). \tag{2.1}$$

Then, the risk contribution of each business line ($X_j, j = 1, \ldots, n$) in the total risk should be determined. In the approach proposed by Panjer (2002), the contribution of the jth line of business is defined as

$$\text{TCE}_{X_i|S}(1-\alpha) = E\left[X_j | S > S_{1-\alpha}\right]. \tag{2.2}$$

The formula above is based on the additivity property of expected values. We call Panjer method the decomposition approach. It is obvious that $E\left[X_j|S > S_{1-\alpha}\right] \neq E\left[X_j|X_j > S_{1-\alpha}\right]$. Eq. (2.2) can be expanded as follows:

$$E\left[X_j|S > S_{1-\alpha}\right] = \int_{x_j \epsilon D_{x_j}} x_j f_{X_j|S>S_{1-\alpha}}(x_j) dx_j = \frac{\int_{s>s_{1-\alpha}} \int_{x_j \epsilon D_{x_j}} x_j f_{X_j,S}(x_j, s) dx_j ds}{1-\alpha}, \tag{2.3}$$

where D_{x_j} is the domain of x_j and $f_X(\cdot)$ is the probability density function of X. In Eq. (2.3), the joint distribution of the X_j and S, ($f_{X_j,\sum_{i=1}^{n} X_i = S}(x_j, s)$), or an estimation of that, is required. The TCE for the univariate and multivariate normal family has been well developed in Panjer (2002). In the decomposition approach, the risk contribution of each variable for a multivariate normal distribution is simplified to

$$\text{TCE}_{X_j|S}(1-\alpha) = \mu_j + \sigma_j \rho_{X_j,S}\left(\frac{\varphi(z_{1-\alpha})}{1-\Phi(z_{1-\alpha})}\right), \tag{2.4}$$

where $z_\alpha = (S_\alpha - \mu_S)/\sigma_S$, φ and Φ are the standard normal density and cumulative function, μ_j and σ_j are the mean and the standard deviation of X_j, and $\rho_{X_j,S}$ represents the correlation coefficient between X_j and S. Also $\mu_S = \sum_{j=1}^{n} \mu_j$ and $\sigma_S^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}$ where the $\sigma_{i,j}$ are the elements of the variance–covariance matrix.