Statistics and Probability Letters 83 (2013) 2242–2247

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Large deviations of Shepp statistics for fractional Brownian motion

Enkelejd Hashorva [a], Zhongquan Tan [b,*]

[a] Faculty of Business and Economics, University of Lausanne, Extranef, UNIL-Dorigny, 1015 Lausanne, Switzerland

[b] College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing 314001, PR China

ARTICLE INFO

Article history:
Received 11 May 2013
Received in revised form 10 June 2013
Accepted 11 June 2013
Available online 19 June 2013

MSC:
primary 60G15
secondary 60G70

Keywords:
Shepp statistics
Scan statistics
Exact asymptotics
Extremes
Fractional Brownian motion

ABSTRACT

Define the incremental fractional Brownian field $Z_H(\tau, s) = B_H(s+\tau) - B_H(s)$, $H \in (0, 1)$, where $B_H(s)$ is a standard fractional Brownian motion with Hurst index $H \in (0, 1)$. In this paper we derive the exact asymptotic behaviour of the maximum $M_H(T) = \max_{(\tau,s)\in[0,1]\times[0,T]} Z_H(\tau, s)$ for any $H \in (0, 1/2)$ complimenting thus the result of Zholud (2008) which establishes the exact tail asymptotic behaviour of $M_{1/2}(T)$.

1. Introduction

Let $\{B_H(t), t \geq 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ which is a centred H-self-similar Gaussian process with stationary increments, almost surely continuous sample paths, $B_H(0) = 0$ and its covariance function is given by

$$\mathrm{Cov}(B_H(t), B_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H}), \quad t, s \geq 0.$$

An important random field defined in terms of this fBm is the so-called incremental fractional Brownian motion

$$Z_H(\tau, s) = B_H(s+\tau) - B_H(s), \quad s, \tau \geq 0.$$

In various statistical applications the incremental fractional Brownian motion appears as the limit model. Typically, when independent and identical observations are modelled, then the limit model has $H = 1/2$. A closely related random field, namely the standardised incremental fractional Brownian motion

$$Z_H^*(\tau, s) = \frac{B_H(s+\tau) - B_H(s)}{\tau^H}, \quad s, \tau > 0$$

* Corresponding author. Tel.: +86 13957395469.
E-mail address: tzq728@163.com (Z. Tan).

0167-7152/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.spl.2013.06.013

serves also in various applications as a limit model. For instance with motivation from queuing theory, consider $\{K(t), t \geq 0\}$ a homogeneous Poisson process with $\mathbb{E}\{K(t)\} = \lambda t, \lambda > 0$ and set for some T, τ positive $K(\tau, T) := \sup_{0\leq s\leq T}(K(s+\tau) - K(s))$. The random variable $K(\tau, T)$ is the maximum service length of an $M/G/\infty$ queue with deterministic service time τ; it is also the version of scan statistics on the positive half line (see e.g., Cressie (1980)). For the study of $K(\tau, T)$ the following convergence in distribution

$$\frac{K(\tau, T) - \lambda\tau}{\sqrt{\lambda\tau}} \to \sup_{0\leq s\leq T} Z_{1/2}{}^*(\tau, s), \quad \lambda \to \infty$$

is important since the distribution function of $\sup_{0\leq s\leq T} Z_{1/2}{}^*(\tau, s)$ is derived in Shepp (1971), see also Slepian (1961), Shepp (1966) and Theorem 3.2 in Cressie (1980).

Various authors refer to the process $\{Z_H^*(\tau, T), \tau \geq 0\}$ as the standardised Shepp statistics. Important results for Shepp statistics and related quantities can be found in Deheuvels and Devroye (1987), Siegmund and Venkatraman (1995), Dümbgen and Spokoiny (2001), Kabluchko and Munk (2008) and Zholud (2009).

The recent papers Zholud (2008) and Kabluchko (2007, 2011a) present asymptotic results on the extremes of Shepp statistics and standardised Shepp statistics, i.e., therein the tail asymptotic behaviour of

$$M_H(T) = \sup_{\tau\in[0,1],0\leq s\leq T} Z_H(\tau, s) \quad \text{and} \quad M_H^*(a, b, T) = \sup_{\tau\in[a,b],0\leq s\leq T} Z_H^*(\tau, s), \quad 0 < a < b < \infty$$

for the case $H = 1/2$ is investigated dealing thus with the increments of the Brownian motion.

In view of Zholud (2008) for any $T > 0$

$$P\left(M_{1/2}(T) > u\right) = \widetilde{\mathcal{H}}_* T u^2 \Psi(u)(1 + o(1)), \quad u \to \infty, \tag{1}$$

where Ψ is the survival function of a $N(0, 1)$ random variable and the constant $\widetilde{\mathcal{H}}_*$ is given by

$$\widetilde{\mathcal{H}}_* = \lim_{a\to\infty} \lim_{b\to\infty} a^{-1} e^{-\frac{a+b}{2}} \mathbb{E}\left\{\exp\left(\max_{\substack{0\leq t\leq a\\0\leq s\leq b}} B_{1/2}(t+s+a) - B_{1/2}(t)\right)\right\}.$$

For any Hurst index $H \neq 1/2$ the independence of the increments of B_H does not hold, which has been the crucial property in the derivation of (1). In our main result given in Theorem 2.1 we derive the exact asymptotics of $M_H(T)$ for $H \in (0, 1/2)$, which is the well-known short-range dependence case for fBm. If $H \in (1/2, 1)$, thus we have a long-range dependence, we have a much more involved problem which will be therefore considered elsewhere.

Numerous authors have considered properties and characterisations of fBm, see e.g., Mishura and Valkeila (2011), Kabluchko (2011b) and the references therein. Our contribution present a new result for the Shepp statistics of fBm, which we believe is important for both future theoretical and applied developments.

Clearly, our result on the tail asymptotic behaviour of $M_H(T)$ implies certain asymptotic bounds for the tail asymptotics of $M_H^*(a, b, T)$. The exact asymptotics of $M_H^*(a, b, T)$ is however easier to deal with and follows by a direct application of the results of Chan and Lai (2006) since the pertaining random field is locally stationary; see Mikhaleva and Piterbarg (1996), and Piterbarg (1996) for the main findings concerning locally stationary random fields.

Brief outline of the rest of the paper: Section 2 displays the main result, its proof is given in Section 3.

2. Main result

In the asymptotic theory of Gaussian processes two important constants are crucial, namely the Pickands and Piterbarg constants. Since in our results only the former constant appears, we briefly mention that it is defined by (see Pickands (1969); Piterbarg (1996))

$$\mathcal{H}_{2H} = \lim_{\lambda\to\infty} \lambda^{-1} \mathbb{E}\left\{\exp\left(\max_{t\in[0,\lambda]}\left(\sqrt{2}B_H(t) - t^{2H}\right)\right)\right\} \in (0, \infty).$$

It is well-known that $\mathcal{H}_1 = 1$ and $\mathcal{H}_2 = 1/\sqrt{\pi}$; see Piterbarg (1972) which gives the first rigorous proof of Pickands' theorem presented in Pickands (1969), Dębicki (2002), Wu (2007), Dębicki and Kisowski (2009) and Dębicki and Tabiś (2011) for generalisations of Pickands' constant.

The result of (1) is of some importance for dealing with the general case $H \neq 1/2$. However we cannot use the method of proof in Zholud (2008) which relies on the independence of increments of Brownian motion. Our proof of the main result presented below is strongly motivated by the method utilised in the seminal contribution Piterbarg (2001).

Theorem 2.1. *For any $H \in (0, 1/2)$ and any $T > 0$*

$$P\left(M_H(T) > u\right) = \frac{T}{H}\left(\frac{1}{2}\right)^{1/H} \mathcal{H}_{2H}^2 u^{\frac{2}{H}-2} \Psi(u)(1 + o(1)) \tag{2}$$

holds as $u \to \infty$.