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A quasi-infinitely divisible characteristic function and its exponentiation

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ABSTRACT

We give a quasi-infinitely divisible characteristic function $g(t)$ such that $(g(t))^u$ is not a characteristic function for any $u \in \mathbb{R}$ except for non-negative integers.

1. Introduction and statement of main result

1.1. Quasi-infinitely divisible distributions

Let $ID(\mathbb{R})$ be the class of all infinitely divisible distributions on $\mathbb{R}$, and let $\widehat{\mu}(t) := \int_{\mathbb{R}} e^{itx}\mu(dx)$, $t \in \mathbb{R}$, be the characteristic function of a distribution μ. The following Lévy–Khintchine representation is well-known (see e.g. Sato (1999)). If $\mu \in ID(\mathbb{R})$, then

$$\widehat{\mu}(t) = \exp\left[-\frac{a}{2}t^2 + i\gamma t + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{itx}{1+|x|^2}\right)\nu(dx)\right], \tag{1.1}$$

where $a \geq 0$, $\gamma \in \mathbb{R}$, and ν is a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(dx) < \infty$.

Lindner and Sato (2011) extended stationary distributions of a generalized Ornstein–Uhlenbeck process by defining a sequence of bivariate Lévy processes (see Lindner and Sato (2009, 2011)). What was interesting in their results in Lindner and Sato (2011) is that there appear non-infinitely divisible distributions whose characteristic functions are the quotients of two infinitely divisible characteristic functions. That class is called the class of *quasi-infinitely divisible distributions* and is defined as follows.

Definition 1.1 (*Quasi-Infinitely Divisible Distribution*). A distribution μ on $\mathbb{R}$ is called quasi-infinitely divisible if it has a form of (1.1) with $a \in \mathbb{R}$ and the corresponding measure ν is a signed measure on $\mathbb{R}$ with total variation measure $|\nu|$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)|\nu|(dx) < \infty$.

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It should be noted that the triplet (a, ν, γ) in this case is also unique if each component exists and that infinitely divisible distributions on $\mathbb{R}$ are quasi-infinitely divisible if and only if $a \geq 0$ and the negative part of ν in the Jordan decomposition equals zero. The measure ν is called *quasi*-Lévy measure which appeared in some books (see for example Gnedenko and Kolmogorov (1968, p. 81) or Sato (in press, Section 2.4)). Recently, Aoyama and Nakamura (in press-a) treated some multivariable finite Euler products and showed how they behave in view whether their corresponding normalized functions to be infinitely or quasi-infinitely divisible characteristic functions on $\mathbb{R}^2$.

1.2. Statement of main result

For any non-infinitely but quasi-infinitely divisible characteristic function $g(t)$, there exists $n \in \mathbb{N}$ such that $(g(t))^{1/n}$ is not a characteristic function since if such $n \in \mathbb{N}$ did not exist, then $g(t)$ were infinitely divisible. The following problem has been proposed by Professor Ken-iti Sato. "Find a quasi-infinitely divisible characteristic function $g(t)$ and determine all $0 < u$ for which $(g(t))^u$ is not a characteristic function". This is a special case of the following open problem written in Sato (in press, p. 11), "to find a necessary and sufficient condition for a signed measure to be a quasi-Levy measure".

In this paper, we give an answer of the problem above by using the generalized Dirichlet L distributions defined in Section 2.1. Let p be a prime number and put

$$g(t) := \frac{1 + p^{-\sigma - \mathrm{i}t}}{1 + p^{-\sigma}}, \quad \sigma > 0.$$

The function $g(t)$ can be expressed by $g(t) = 1 + f(t)$, where $f(t) := (p^{-\sigma-\mathrm{i}t} - p^{-\sigma})(1 + p^{-\sigma})^{-1}$. One has $|f(t)| < 1$ by the assumption $\sigma > 0$. Therefore $(g(t))^u$ is uniquely defined by the binomial series for any $u \in \mathbb{R}$.

Let $\mathbb{N}_0$ be the set of non-negative integers. Then we have the following.

Theorem 1.2. *The function $g(t)$ is a not infinitely divisible but quasi-infinitely divisible characteristic function and $(g(t))^u$ is not a characteristic function for any $\sigma > 0$ and $u \in \mathbb{R} \setminus \mathbb{N}_0$.*

2. Proof of main result

2.1. Generalized Dirichlet L distributions

The Riemann zeta function is a function of a complex variable $s = \sigma + \mathrm{i}t$, for $\sigma > 1$ given by $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$. The series is called the Dirichlet series and converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. Put $Z_\sigma(t) := \zeta(\sigma + \mathrm{i}t)/\zeta(\sigma)$, $t \in \mathbb{R}$, then $Z_\sigma(t)$ is known to be a characteristic function. A distribution μ_σ on $\mathbb{R}$ is said to be a Riemann zeta distribution with parameter σ if it has $Z_\sigma(t)$ as its characteristic function. The Riemann zeta distribution is known to be infinitely divisible (see e.g. Gnedenko and Kolmogorov (1968, p. 75)).

Let $M > 0$, $\sigma, t \in \mathbb{R}$ and $s = \sigma + \mathrm{i}t$. As a generalization of the Riemann zeta function, we consider the following series

$$D(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{where } -M \leq a(n) \leq M. \tag{2.1}$$

We call this series a generalized Dirichlet L series (see e.g. Apostol (1976, p. 224)). When $\sigma > 1$, the series converges absolutely for any $-M \leq a(n) \leq M$. In the present paper, we treat only the case $a(n) \neq 0$ for some $n \in \mathbb{N}$, and the series of $D(s)$ converges absolutely.

Now we introduce probability distributions on $\mathbb{R}$ produced by the generalized Dirichlet L series $D(s)$ defined by (2.1).

Definition 2.1 (*Generalized Dirichlet L Distribution*)**.** Suppose $0 \leq a(n) \leq M$ for any $n \in \mathbb{N}$ or $-M \leq a(n) \leq 0$ for any $n \in \mathbb{N}$. Then we define a generalized Dirichlet L random variable X_σ with probability distribution on $\mathbb{R}$ given by

$$\Pr(X_\sigma = -\log n) = \frac{a(n) n^{-\sigma}}{D(\sigma)}.$$

Note that these distributions belong to a special case of multidimensional Shintani zeta distribution defined by Aoyama and Nakamura (in press-b). It is easy to see that these distributions are probability distributions (see also Lemma 2.2) since $a(n)n^{-\sigma}/D(\sigma) \geq 0$ for each $n \in \mathbb{N}$ when $0 \leq a(n) \leq M$ for any $n \in \mathbb{N}$ or $-M \leq a(n) \leq 0$ for any $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} \frac{a(n)n^{-\sigma}}{D(\sigma)} = \frac{1}{D(\sigma)} \sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma} = \frac{D(\sigma)}{D(\sigma)} = 1.$$

2.2. Proof

To prove the main results, we need the following lemmas.