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Tempered fractional Brownian motion

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ABSTRACT

Tempered fractional Brownian motion (TFBM) modifies the power law kernel in the moving average representation of a fractional Brownian motion, adding an exponential tempering. Tempered fractional Gaussian noise (TFGN), the increments of TFBM, form a stationary time series that can exhibit semi-long range dependence. This paper develops the basic theory of TFBM, including moving average and spectral representations, sample path properties, and an application to modeling wind speed.

1. Introduction

This paper defines a new stochastic process, which we call tempered fractional Brownian motion (TFBM), defined by exponentially tempering the power law kernel in the moving average representation of a fractional Brownian motion (FBM). The stationary increments of TFBM are called tempered fractional Gaussian noise (TFGN). When FGN is long range dependent, the corresponding TFGN exhibits semi-long range dependence: Its autocovariance function closely resembles that of FGN on an intermediate scale, but eventually falls off more rapidly. The spectral density of TFGN resembles a negative power law for low frequencies, but remains bounded at very low frequencies.

2. Moving average representation

Let $\{B(t)\}_{t\in\mathbb{R}}$ be a real-valued Brownian motion on the real line, a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $\sigma^2|t|$ for all $t \in \mathbb{R}$, for some $\sigma > 0$. Define an independently scattered Gaussian random measure $B(dx)$ with control measure $m(dx) = \sigma^2 dx$ by setting $B[a, b] = B(b) - B(a)$ for any real numbers $a < b$, and then extending to all Borel sets. Then the stochastic integrals $I(f) := \int f(x)B(dx)$ are defined for all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int f(x)^2 dx < \infty$, as Gaussian random variables with mean zero and covariance $\mathbb{E}[I(f)I(g)] = \sigma^2 \int f(x)g(x)dx$, see for example Chapter 3 in Samorodnitsky and Taqqu (1994).

Definition 2.1. Given an independently scattered Gaussian random measure $B(dx)$ on $\mathbb{R}$ with control measure $\sigma^2 dx$, for any $\alpha < \frac{1}{2}$ and $\lambda \geq 0$, the stochastic integral

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$$B_{\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha}\right] B(dx) \tag{2.1}$$

where $(x)_+ = xI(x > 0)$, and $0^0 = 0$, will be called a *tempered fractional Brownian motion* (TFBM).

It is easy to check that the function

$$g_{\alpha,\lambda,t}(x) := e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \tag{2.2}$$

is square integrable over the entire real line for any $\alpha < \frac{1}{2}$, so that TFBM is well-defined. When $-1/2 < \alpha < 1/2$, FBM is a special case of TFBM with $\lambda = 0$. Note also that

$$g_{\alpha,\lambda,ct}(cx) = c^{-\alpha} g_{\alpha,c\lambda,t}(x) \tag{2.3}$$

for all $t, x \in \mathbb{R}$ and all $c > 0$. The next results shows that TFBM has a nice scaling property, involving both the time scale and the tempering. Here the symbol $\triangleq$ indicates equality of finite dimensional distributions.

Proposition 2.2. *TFBM (2.1) is a Gaussian stochastic process with stationary increments, such that*

$$\left\{B_{\alpha,\lambda}(ct)\right\}_{t\in\mathbb{R}} \triangleq \left\{c^H B_{\alpha,c\lambda}(t)\right\}_{t\in\mathbb{R}} \tag{2.4}$$

for any scale factor $c > 0$, where the Hurst index $H = 1/2 - \alpha$.

Proof. Since $B(dx)$ has control measure $m(dx) = \sigma^2 dx$, the random measure $B(c\,dx)$ has control measure $c^{1/2}\sigma^2 dx$. Given $t_1 < t_2 < \cdots < t_n$, a change of variable $x = cx'$ then yields

$$\begin{aligned}\left(B_{\alpha,\lambda}(ct_i) : i = 1, \ldots, n\right) &= \left(\int g_{\alpha,\lambda,ct_i}(x)B(dx) : i = 1, \ldots, n\right)\\ &\stackrel{d}{=} \left(\int c^{-\alpha} g_{\alpha,c\lambda,t_i}(x')c^{1/2}B(dx') : i = 1, \ldots, n\right)\end{aligned}$$

so that (2.4) holds with $H = 1/2 - \alpha$. For any $s, t \in \mathbb{R}$, the integrand (2.2) satisfies $g_{\alpha,\lambda,s+t}(s+x) - g_{\alpha,\lambda,s}(s+x) = g_{\alpha,\lambda,t}(x)$, and hence a change of variable $x = s + x'$ in the moving average representation yields

$$\left(B_{\alpha,\lambda}(s+t_i) - B_{\alpha,\lambda}(s) : i = 1, \ldots, n\right) \triangleq \left(\int g_{\alpha,\lambda,t_i}(x')B(dx') : i = 1, \ldots, n\right)$$

which shows that TFBM has stationary increments. □

Proposition 2.3. *TFBM (2.1) has the covariance function*

$$\mathrm{Cov}\left[B_{\alpha,\lambda}(t), B_{\alpha,\lambda}(s)\right] = \frac{\sigma^2}{2}\left[C_t^2\,|t|^{2H} + C_s^2\,|s|^{2H} - C_{t-s}^2\,|t-s|^{2H}\right] \tag{2.5}$$

for any $s, t \in \mathbb{R}$, where $H = 1/2 - \alpha$. Here

$$C_t^2 = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma\left(H+\frac{1}{2}\right)}{\sqrt{\pi}}\frac{1}{(2\lambda|t|)^H}K_H(\lambda|t|), \tag{2.6}$$

for $t \neq 0$, where $K_\nu(z)$ is the modified Bessel function of the second kind, and $C_0^2 = 0$.

Proof. Use the moving average representation (2.1) with $\sigma = 1$ to define

$$\begin{aligned}C_t^2 := \mathbb{E}[B_{\alpha,\lambda|t|}(1)^2] &= \int_{-\infty}^{+\infty}\left[e^{-\lambda t(1-x)_+}(1-x)_+^{-\alpha} - e^{-\lambda t(-x)_+}(-x)_+^{-\alpha}\right]^2 dx\\ &= \int_{-\infty}^{+\infty} e^{-2\lambda t(1-x)_+}(1-x)_+^{-2\alpha}dx + \int_{-\infty}^{+\infty} e^{-2\lambda t(-x)_+}(-x)_+^{-2\alpha}dx\\ &\quad -2\int_{-\infty}^{+\infty} e^{-\lambda t(1-x)_+}(1-x)_+^{-\alpha}e^{-\lambda t(-x)_+}(-x)_+^{-\alpha}dx.\end{aligned} \tag{2.7}$$

Apply the definition of the gamma function, along with a standard integral formula from p. 344 in Gradshteyn and Ryzhik (2000), to see that (2.6) holds. Since TFBM has stationary increments, it follows from (2.4) that $\mathbb{E}[B_{\alpha,\lambda}(t)^2] = |t|^{2H}C_t^2$ for all t real. Recall the elementary formula $ab = \frac{1}{2}[a^2 + b^2 - (a-b)^2]$, set $a = B^{\alpha,\lambda}(t)$ and $b = B^{\alpha,\lambda}(s)$, take expectations, and use the stationary increments property again, to see that (2.5) holds. □

Remark 2.4. The integral representation (2.1) is causal, i.e., $B_{\alpha,\lambda}(t)$ depends only on the values of $B(s)$ for $s \le t$. For applications to spatial statistics, consider