Statistics and Probability Letters 83 (2013) 2291–2296

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On weak invariance principles for sums of dependent random functionals

Moritz Jirak

Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6 D-10099 Berlin, Germany

ARTICLE INFO

Article history:
Received 6 April 2013
Received in revised form 12 June 2013
Accepted 12 June 2013
Available online 20 June 2013

MSC:
60F17
60F25
62G10

Keywords:
Weak invariance principle
Weakly dependent functionals

ABSTRACT

Given a sequence of random functionals $\{X_k(u)\}_{k\in\mathbb{Z}} \in \mathbb{L}^2[0,1]$, the normalized partial sum-process $S_n(t,u) = n^{-1/2}\big(X_1(u) + \cdots + X_{\lfloor nt\rfloor}(u)\big)$, $t, u \in [0,1]$ is considered. Given two moments and a fairly general dependence structure, a weak invariance principle is established, extending a recent result of Berkes et al. (2013).

1. Introduction and main results

Let $\{X_k(u)\}_{k\in\mathbb{Z}}$ be a sequence of random functions, square integrable on [0, 1]. For $p \geq 1$, denote with $\big|.\big|_{\mathbb{L}^p}$ the $\mathbb{L}^p[0,1]$-norm, and put $\big\|.\big\|_p = \big(\mathbb{E}\big[|.|^p\big]\big)^{1/p}$. Let $\mathbb{S}$ be some measurable space. Given a sequence $\{\epsilon_k\}_{k\in\mathbb{Z}} \in \mathbb{S}$ of independent and identically distributed random variables, we consider the random functions

$$X_k(u) = g\big(\epsilon_k, \epsilon_{k-1}, \ldots\big)(u), \tag{1.1}$$

where $g\big(.\big)$ is some measurable function such that $\{X_k(u)\}_{k\in\mathbb{Z}}$ is well-defined in $\mathbb{L}^2[0,1]$. If it goes without confusion, we will write g for a function $g(u)$, hence X_k for $X_k(u)$. For convenience, we will also write $g\big(\xi_k\big)(u)$, with $\xi_k = (\epsilon_k, \epsilon_{k-1}, \ldots)$. Following Wu (2005), let $\{\epsilon_k'\}_{k\in\mathbb{Z}}$ be an independent copy of $\{\epsilon_k\}_{k\in\mathbb{Z}}$ on the same probability space, and define the 'filters' $\xi_k^{(m,')}, \xi_k^{(m,*)}$ as

$$\xi_k^{(m,')} = (\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_{k-m}', \epsilon_{k-m-1}, \ldots) \quad \text{and} \quad \xi_k^{(m,*)} = (\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_{k-m}, \epsilon_{k-m-1}', \epsilon_{k-m-2}' \ldots). \tag{1.2}$$

We put $\xi_k' = \xi_k^{(k,')} = (\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_0', \epsilon_{-1}, \ldots)$ and $\xi_k^* = \xi_k^{(k,*)} = (\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_0, \epsilon_{-1}', \epsilon_{-2}' \ldots)$. In analogy, we put $X_k^{(m,')} = g\big(\xi_k^{(m,')}\big)$ and $X_k^{(m,*)} = g\big(\xi_k^{(m,*)}\big)$, in particular we have $X_k' = X_k^{(k,')}$ and $X_k^* = X_k^{(k,*)}$. Note that the representation $X_k = g\big(\xi_k\big)$ implies that $\{X_k\}_{k\in\mathbb{Z}}$ is stationary and ergodic. We will derive our results under the following assumptions.

E-mail address: m0ritz@yahoo.com.

0167-7152/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.spl.2013.06.014

Assumption 1.1. The sequence $\{X_k(u)\}_{k\in\mathbb{Z}}$ satisfies

(i) $\mathbb{E}\big[X_0(u)\big] = 0$ for almost all $u \in [0, 1]$,

(ii) $\sum_{k=0}^{\infty}\Big|\,\|X_k - X_k'\|_2\Big|_{\mathbb{L}^2} < \infty$ or (ii*) $\sum_{k=0}^{\infty}\Big|\,\|X_k - X_k^*\|_2\Big|_{\mathbb{L}^2} < \infty$.

A discussion on the assumptions and a comparison of relevant results from the literature are given below Theorem 1.2. We show in Section 2 (cf. Lemma 2.5) that the series

$$\Gamma(u, v) = \mathbb{E}\big[X_0(u)X_0(v)\big] + \sum_{k=1}^{\infty}\mathbb{E}\big[X_k(u)X_0(v)\big] + \sum_{k=1}^{\infty}\mathbb{E}\big[X_k(v)X_0(u)\big]$$

is convergent in $\mathbb{L}^2[0, 1]^2$. $\Gamma(u, v)$ is positive definite, which implies the following expansion (see for instance Chapter 4 in Indritz, 1963): $\Gamma(u, v) = \sum_{l=1}^{\infty}\lambda_l \nu_l(u)\nu_l(v)$, where $\lambda_l \geq 0$, ν_l, $l = 1, \ldots$ are the eigenvalues and eigenfunctions respectively of the covariance operator $\Gamma(u, v)$. Routine arguments (cf. Berkes et al., 2013) then yield that one may define the Gaussian process

$$\mathcal{G}(t, u) = \sum_{l=1}^{\infty}\sqrt{\lambda_l}\nu_l(u)\mathcal{W}_l(t), \tag{1.3}$$

where $\{\mathcal{W}_l(t)\}_{1\leq l<\infty}$ is a sequence of independent Wiener processes. This expansion is often referred to as the Karhunen–Loéve expansion.

Theorem 1.2. *Suppose that Assumption 1.1 holds, and let* $S_n(t, u) = n^{-1/2}\sum_{k=1}^{\lceil nt\rceil}X_k(u)$. *Then for every n, we can define a Gaussian process* $\mathcal{G}_n(t, u)$ *such that*

$$\big\{\mathcal{G}_n(t, u),\ t, u \in [0, 1]\big\} \stackrel{d}{=} \big\{\mathcal{G}(t, u),\ t, u \in [0, 1]\big\} \quad \textit{and}$$

$$\sup_{0\leq t\leq 1}\int_0^1\Big(\mathcal{G}_n(t, u) - S_n(t, u)\Big)^2 du = \mathcal{O}_P\big(1\big).$$

As an immediate consequence, we obtain the following corollary.

Corollary 1.3. *Let* $\mathbb{S}$ *be a space endowed with the (pseudo)-metric* $\sup\sqrt{\int|.|^2}$, *rich enough such that* $S_n(t, u) \in \mathbb{S}$ *for each n. Suppose that Assumption 1.1 holds. Then* $S_n(t, u)$ *converges weakly to* $\mathcal{G}(t, u)$, *given in (1.3), where the weak convergence is to be understood in the Hoffmann–Jørgensen sense.*

The topic of CLT for the sum of random processes is widely studied in the literature. For some applications in terms of functional data analysis, see Horváth and Kokoszka (2012) and Hörmann and Kokoszka (2010). In the case of dependent random variables, sharp results have been obtained under various mixing and martingale approximation techniques in Hilbert spaces, see Dedecker and Merlevède (2003) and the references therein. However, verifying mixing conditions is generally not easy and without additional continuity conditions, even AR(1) processes may fail to be strong mixing (see Andrews, 1984). Therefore Berkes et al. (2013) used an alternative approach based on m-dependent approximations. In Hörmann and Kokoszka (2010) it is shown that such conditions, based on (1.2), can be easily verified for many processes such as functional ARMA or GARCH models. Berkes et al. (2013) derived their results under the condition

$$\text{(iii}^*\text{)}\ \sum_{k=0}^{\infty}\Big(\mathbb{E}\big[\big|X_k(u) - X_k^*(u)\big|_{\mathbb{L}^2}^p\big]\Big)^{1/q} < \infty,$$

where $2 < p < q$. Note that (iii*) requires the existence of more moments compared to Assumption 1.1, and in most cases we have $\|X_k(u) - X_k^*(u)\|_p > \|X_k(u) - X_k'(u)\|_p$. To exemplify this claim, let us consider the linear process $\sum_{j=0}^{\infty}a_j\big(\epsilon_{k-j}(u)\big)$ with IID innovations $\{\epsilon_k(u)\}_{k\in\mathbb{Z}}$, where $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of bounded, linear operators. Assumption 1.1 is valid if $\mathbb{E}\big[\epsilon_k(u)\big] = 0$ for $u \in [0, 1]$, $\big|\|\epsilon_k(u)\|_2\big|_{\mathbb{L}^2} < \infty$ and $\sum_{j=0}^{\infty}\big|a_j\big|_{\mathbf{L}(\mathbb{L}^2)} < \infty$, where $\big|.\big|_{\mathbf{L}(\mathbb{L}^p)}$ denotes the usual operator norm for $p \geq 1$. Contrary, condition (iii*) is valid if for $2 < p < q$

$$\sum_{l=0}^{\infty}\Bigg(\mathbb{E}\Bigg[\Bigg|\sum_{j>l}a_j\big(\epsilon_{l-j}(u)\big)\Bigg|_{\mathbb{L}^2}^p\Bigg]\Bigg)^{1/q}. \tag{1.4}$$

To simplify matters further, suppose that $a_j\big(\epsilon_{k-j}(u)\big) = a_j\epsilon_{k-j}(u)$ for $a_j \in \mathbb{R}$, where we assume that $|a_j| = \mathcal{O}\big(j^{-\beta}\big)$, $\beta > 1$. Then Jensen's inequality implies that (iii*) is only valid if

$$\sum_{l=0}^{\infty}\Bigg(\mathbb{E}\Bigg[\Bigg|\sum_{j>l}a_j\epsilon_{l-j}\Bigg|_{\mathbb{L}^2}\Bigg]\Bigg)^{p/q} < \infty. \tag{1.5}$$