Statistics and Probability Letters 83 (2013) 2297–2307

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On a mixture representation of the conditional inactivity time of a coherent system

Shuhong Zhang [a], Xiuying Feng [a], Peng Zhao [b,*]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

[b] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China

ARTICLE INFO

Article history:
Received 10 May 2013
Received in revised form 13 June 2013
Accepted 14 June 2013
Available online 21 June 2013

Keywords:
Coherent system
Order statistics
Inactivity time
Signature
Stochastic order

ABSTRACT

In this paper, we provide a mixture representation of the reliability function of the conditional inactivity time of a coherent system in terms of the reliability functions of conditional inactivity times of order statistics. Based on this representation, we then carry out stochastic comparisons between the conditional inactivity times of two coherent systems having either different structures or different component lifetimes. The results established here strengthen and extend some results known in the literature.

1. Introduction

As a dual notion to the residual lifetime, the inactivity time plays an important role in both reliability and survival analysis. For a reliability system with lifetime T, which can be regarded as a black box wherein the exact failure times of its components cannot be observed, it is of great interest for engineers and reliability analysts to get the knowledge on stochastic properties of the inactivity time $[t - T | T \le t]$, the time elapsed since the failure of the system, so that they could choose a reasonable epoch to initiate preventive maintenance or replacement of the whole system. Likewise, stochastic comparison on the inactivity time will be helpful to get a more reliable design of the system in decision-making. This notion has a close connection with so called autopsy data that can be viewed as the information obtained by examining the statuses of the components of a failed system. In the literature, many researchers have dealt with the inactivity time of a component or a system; see, for example, Navarro et al. (1997), Asadi (2006), Khaledi and Shaked (2007), Li and Zhang (2008), Li and Zhao (2006, 2008), Tavangar and Asadi (2010), Zhao and Balarishnan (2009) and Zhao et al. (2008).

Consider a coherent system of order n (see Barlow and Proschan, 1981) whose independent and identically distributed (i.i.d.) component lifetimes $X_1, \ldots, X_n$ are arising from a common continuous distribution F. Let $\tau(\boldsymbol{X}) = \tau(X_1, \ldots, X_n)$ denote the lifetime of the coherent system, where τ is a coherent structure function. Samaniego (1985) (see also Kochar et al., 1999) indicated that the reliability function of $\tau(\boldsymbol{X})$ can be written as a mixture of reliability functions of order statistics, i.e.,

$$\mathsf{P}(\tau(\boldsymbol{X}) > t) = \sum_{i=1}^{n} p_i \, \mathsf{P}(X_{i:n} > t),$$

* Corresponding author. Tel.: +86 51683500332.
E-mail addresses: zhaop@lzu.edu.cn, zhaop07@gmail.com (P. Zhao).

0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.06.017

where $\boldsymbol{p} = (p_1, \ldots, p_n)$ is the system's *signature* with $p_i = \mathsf{P}(\tau(\boldsymbol{X}) = X_{i:n})$ and $X_{i:n}$ is the ith smallest order statistic among $X_1, \ldots, X_n$. The signature of a system does not depend on the distribution of $X_1, \ldots, X_n$. For more details on this topic, one may refer to Boland (2001), Samaniego (2007), Triantafyllou and Koutras (2008), Samaniego et al. (2009), Navarro et al. (2010), Mahmoudi and Asadi (2011), etc. The mixture representation of the reliability function of the coherent system has been found to be useful to obtain stochastic comparison results and many researchers have discussed this topic; see, for example, Navarro et al. (2005), Li and Zhang (2008), Navarro et al. (2008), Zhang (2010b), Eryılmaz (2011a,b) and Burkschat and Navarro (2013).

Let us first recall some pertinent stochastic orderings that will be used in this paper. Let X and Y be two nonnegative random variables with distribution functions F and G, reliability functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, density functions f and g, respectively. X is said to be smaller than Y in the (i) *usual stochastic order* (denoted by $X \le_{\rm st} Y$) if $\overline{G}(x) \ge \overline{F}(x)$ for all x; (ii) *hazard rate order* (denoted by $X \le_{\rm hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in x; (iii) *reversed hazard rate order* (denoted by $X \le_{\rm rh} Y$) if $G(x)/F(x)$ is increasing in x; (iv) *likelihood ratio order* (denoted by $X \le_{\rm lr} Y$) if $g(x)/f(x)$ is increasing in x. For two discrete distributions $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$, $\boldsymbol{p}$ is said to be smaller than $\boldsymbol{q}$ in the (i) usual stochastic order (denoted by $\boldsymbol{p} \le_{\rm st} \boldsymbol{q}$) if $\sum_{j=i}^{n} q_j \ge \sum_{j=i}^{n} p_j$ for all i; (ii) hazard rate order (denoted by $\boldsymbol{p} \le_{\rm hr} \boldsymbol{q}$) if $\sum_{j=i}^{n} q_j / \sum_{j=i}^{n} p_j$ is increasing in i; (iii) reversed hazard rate order (denoted by $\boldsymbol{p} \le_{\rm rh} \boldsymbol{q}$) if $\sum_{j=1}^{i} q_j / \sum_{j=1}^{i} p_j$ is increasing in i; (iv) likelihood ratio order (denoted by $\boldsymbol{p} \le_{\rm lr} \boldsymbol{q}$) if q_i/p_i is increasing in i, when $p_i, q_i > 0$.

Zhang (2010a) expressed the reliability function of inactivity time $[t - \tau(\boldsymbol{X}) \mid \tau(\boldsymbol{X}) \le t]$ of the coherent system as a mixture of reliability functions of inactivity times of order statistics, i.e.,

$$\mathsf{P}(t - \tau(\boldsymbol{X}) > x \mid \tau(\boldsymbol{X}) \le t) = \sum_{i=1}^{n} p_i(t)\, \mathsf{P}(t - X_{i:n} > x \mid X_{i:n} \le t), \tag{1}$$

where $p_i(t) = \mathsf{P}(\tau(\boldsymbol{X}) = X_{i:n} \mid \tau(\boldsymbol{X}) \le t)$ and $\sum_{i=1}^{n} p_i(t) = 1$. Let $\tau_i(\boldsymbol{X})$ be the lifetime of a coherent system with i.i.d. component lifetimes $X_1, \ldots, X_n$ and the corresponding mixing coefficient vector $\boldsymbol{p}_i(t)$, $i = 1, 2$. Zhang (2010a) further proved that

$$\boldsymbol{p}_1(t) \le_{\rm st} (\le_{\rm rh}, \le_{\rm lr})\, \boldsymbol{p}_2(t) \Longrightarrow [t - \tau_1(\boldsymbol{X}) \mid \tau_1(\boldsymbol{X}) \le t] \ge_{\rm st} (\ge_{\rm hr}, \ge_{\rm lr})\, [t - \tau_2(\boldsymbol{X}) \mid \tau_2(\boldsymbol{X}) \le t]. \tag{2}$$

For a failed system, it is unnecessary that all of the components are failed.

In this paper, we focus our attention on the inactivity time of a coherent system with signature $\boldsymbol{p} = (p_1, \ldots, p_n)$ when the system has failed and at least $n-k+1$ components of the system are alive at time t, i.e., $[t - \tau(\boldsymbol{X}) \mid \tau(\boldsymbol{X}) \le t, X_{k:n} > t]$. Then we prove that the reliability function of this kind of conditional inactivity time can also be expressed as a mixture of reliability functions of conditional inactivity times of the corresponding order statistics. Upon using this mixture representation, we present some stochastic comparison results for the conditional inactivity times between two coherent systems having different structures or different component lifetimes. It can be seen that some of the results established here are similar to those in (2).

Throughout this paper, we use the terms *increasing* and *decreasing* in place of *non-decreasing* and *non-increasing*, respectively, and all random variables under consideration have 0 as the common left endpoint of their supports and expectations are finite as they appear.

2. Main results

Consider a coherent system consisting of n components with i.i.d. lifetimes $X_1, \ldots, X_n$ having a continuous distribution function F, and $X_{1:n}, \ldots, X_{n:n}$ are the corresponding order statistics. Let $\overline{F} = 1 - F$ be the common reliability function, $\tau(\boldsymbol{X}) = \tau(X_1, \ldots, X_n)$ be the system's lifetime and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be the system's signature. In this section, we first present a mixture representation for the reliability function of the conditional inactivity time $[t - \tau(\boldsymbol{X}) \mid \tau(\boldsymbol{X}) \le t, X_{k:n} > t]$.

Theorem 1. *Suppose that* $\mathsf{P}(\tau(\boldsymbol{X}) \le t, X_{k:n} > t) > 0$, *for some* $k \in \{2, \ldots, n\}$. *Then, for all* $x \ge 0$,

$$\mathsf{P}(t - \tau(\boldsymbol{X}) > x \mid \tau(\boldsymbol{X}) \le t, X_{k:n} > t) = \sum_{j=1}^{k-1} p_j(t,k)\, \mathsf{P}(t - X_{j:n} > x \mid X_{j:n} \le t, X_{k:n} > t), \tag{3}$$

where $\boldsymbol{p}(t,k) = \big(p_1(t,k), \ldots, p_{k-1}(t,k), 0, \ldots, 0\big)$ *such that*

$$p_j(t,k) = \mathsf{P}(\tau(\boldsymbol{X}) = X_{j:n} \mid \tau(\boldsymbol{X}) \le t, X_{k:n} > t) = \frac{p_j\, \mathsf{P}(X_{j:n} \le t, X_{k:n} > t)}{\sum_{i=1}^{k-1} p_i\, \mathsf{P}(X_{i:n} \le t, X_{k:n} > t)}$$

and $\sum_{j=1}^{k-1} p_j(t,k) = 1$.