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Exponential rate of convergence for some Markov operators

H. Wojewódka *

Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

The exponential rate of convergence for Markov operators is established. The operators correspond to continuous iterated function systems which are a very useful tool in some cell cycle models.

1. Introduction

We are concerned with Markov operators corresponding to continuous iterated function systems. The main purpose of the paper is to prove spectral gap assuring exponential rate of convergence. The operators under consideration were used in Lasota and Mackey (1999), where the authors studied some cell cycle model. See also Tyson and Hannsgen (1988) or Murray and Hunt (1993) to get more details on the subject. Lasota and Mackey proved only stability, while we managed to evaluate rate of convergence, bringing some information important from biological point of view. In our paper we base on coupling methods introduced in Hairer (2002). In the same spirit, exponential rate of convergence was proved in Ślęczka (2011) for classical iterated function systems (see also Hairer and Mattingly (2008) or Kapica and Ślęczka (2012)). It is worth mentioning here that our result will allow us to show the Central Limit Theorem (CLT) and the Law of Iterated Logarithm (LIL). To do this, we will adapt general results recently proved in Bołt et al. (2012) or in Komorowski and Walczuk (2012). The proof of CLT and LIL will be provided in a future paper.

The organization of the paper goes as follows. Section 2 introduces basic notation and definitions that are needed throughout the paper. Most of them are adapted from Billingsley (1968), Meyn and Tweedie (1993), Lasota and Yorke (1994) and Szarek (2003). Biological background is shortly presented in Section 3. Sections 4 and 5 provide the mathematical derivation of the model and the main theorem (Theorem 2), which establishes the exponential rate of convergence in the model. Sections 6–8 are devoted to the construction of coupling measure for iterated function systems. Thanks to the results presented in Section 9 we are finally able to present the proof of the main theorem in Section 10.

2. Notation and basic definitions

Let (X, ϱ) be a Polish space. We denote by B_X the family of all Borel subsets of X. Let $C(X)$ be the space of all bounded and continuous functions $f : X \to R$ with the supremum norm.

* Tel.: +48 695690163.
E-mail addresses: haniawojew@wp.pl, hwojewod@mat.ug.edu.pl.

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We denote by $M(X)$ the family of all Borel measures on X and by $M_{fin}(X)$ and $M_1(X)$ its subfamilies such that $\mu(X) < \infty$ and $\mu(X) = 1$, respectively. Elements of $M_{fin}(X)$ which satisfy $\mu(X) \leq 1$ are called sub-probability measures. To simplify notation, we write

$$\langle f, \mu\rangle = \int_X f(x)\mu(dx) \quad \text{for} f \in C(X), \mu \in M(X).$$

An operator $P : M_{fin}(X) \to M_{fin}(X)$ is called a Markov operator if

(i) $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad$ for $\lambda_1, \lambda_2 \geq 0,\ \mu_1, \mu_2 \in M_{fin}(X)$;
(ii) $P\mu(X) = \mu(X)$ for $\mu \in M_{fin}(X)$.

If, additionally, there exists a linear operator $U : C(X) \to C(X)$ such that

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for} f \in C(X), \mu \in M_{fin}(X),$$

an operator P is called a Feller operator. Every Markov operator P may be extended to the space of signed measures on X denoted by $M_{sig}(X) = \{\mu_1 - \mu_2 :\ \mu_1, \mu_2 \in M_{fin}(X)\}$. For $\mu \in M_{sig}(X)$ we denote by $\|\mu\|$ the total variation norm of μ, i.e.

$$\|\mu\| = \mu^+(X) + \mu^-(X),$$

where μ^+ and μ^- come from the Hahn-Jordan decomposition of μ (see Halmos (1974)). For fixed $\bar{x} \in X$ we also consider the space $M_1^1(X)$ of all probability measures with the first moment finite, i.e. $M_1^1(X) = \{\mu \in M_1(X) : \int_X \varrho(x, \bar{x})\mu(dx) < \infty\}$. The family is idependent of the choice of $\bar{x} \in X$. We call $\mu_* \in M_{fin}(X)$ an invariant measure of P if $P\mu_* = \mu_*$. For $\mu \in M_{fin}(X)$ we define the support of μ by

$$\operatorname{supp}\mu = \{x \in X :\ \mu(B(x, r)) > 0 \quad \text{for} r > 0\},$$

where $B(x, r)$ is the open ball in X with center at $x \in X$ and radius $r > 0$.

In $M_{sig}(X)$ we introduce the Fourtet-Mourier norm

$$\|\mu\|_{\mathcal{L}} = \sup_{f\in\mathcal{L}} |\langle f, \mu\rangle|,$$

where $\mathcal{L} = \{f \in C(X) :\ |f(x) - f(y)| \leq \varrho(x, y),\ |f(x)| \leq 1 \ \text{ for} x, y \in X\}$. The space $M_1(X)$ with the metric $\|\mu_1 - \mu_2\|_{\mathcal{L}}$ is complete (see Fortet and Mourier (1953) or Rachev (1991)). In $M_{sig}^1(X) = \{\mu \in M_{sig}(X) : \int_X \varrho(x, \bar{x})\mu(dx) < \infty\}$ we introduce the Wasserstein norm

$$\|\mu\|_{\mathcal{H}} = \sup_{f\in\mathcal{H}} |\langle f, \mu\rangle|,$$

where $\mathcal{H} = \{f \in C(X) :\ |f(x) - f(y)| \leq \varrho(x, y) \ \text{ for} x, y \in X\}$. Analogously, the space $M_1^1(X)$ with the metric $\|\mu_1 - \mu_2\|_{\mathcal{H}}$ is complete (see Villani (2009)). Note that

$$\|\mu\|_{\mathcal{L}} \leq \|\mu\|_{\mathcal{H}} \quad \text{for} \mu \in M_1^1(X). \tag{2.1}$$

3. Shortly about the model of cell division cycle

Let $(\Omega, \mathcal{F}, \text{Prob})$ be a probability space. Suppose that each cell in a considered population consists of d different substances, whose masses are described by the vector $y(t) = (y^1(t), \ldots, y^d(t))$, where $t \in [0, T]$ denotes an age of a cell. We assume that the evolution of the vector $y(t)$ is given by the formula $y(t) = \Pi(x, t)$, where $\Pi(x, 0) = x$. Here $\Pi : X \times [0, T) \to X$ is a given function. A simple example fulfilling these criteria is given by assuming that $y(t)$ satisfies a system of ordinary differential equations

$$\frac{dy}{dt} = g(t, y) \tag{3.1}$$

with the initial condition $y(0) = x$ and the solution of (3.1) is given by $y(t) = \Pi(x, t)$.

If x_n denotes the initial value $x = y(0)$ of substances in the n-th generation and t_n denotes the mitotic time in the n-th generation, the distribution is given by

$$\text{Prob}(t_n \in I | x_n = x) = \int_I p(x, s)ds \quad \text{for} I \in [0, T], n \in N. \tag{3.2}$$

The vector $y(t_n) = \Pi(x_n, t_n)$ with $y(0) = \Pi(x, 0) = x$ describes an amount of intercellular substance just before cell division in the n-th generation. We assume that each daughter cell contains exactly half of the components of its stem cell. Hence

$$x_{n+1} = \frac{1}{2}\Pi(x_n, t_n) \quad \text{for} n = 0, 1, 2, \ldots. \tag{3.3}$$

The bahaviour of (3.2) and (3.3) may be also described by the sequence $(\mu_n)_{n\geq 1}$ of distributions

$$\mu_n(A) = \text{Prob}(x_n \in A) \quad \text{for} n = 0, 1, 2, \ldots \text{ and} A \in B_X.$$

See Lasota and Mackey (1999) for more details.