Statistics and Probability Letters 83 (2013) 2372–2385

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Berry–Esséen bounds for the least squares estimator for discretely observed fractional Ornstein–Uhlenbeck processes

Khalifa Es-Sebaiy *

National School of Applied Sciences - Marrakesh, Cadi Ayyad University, BP 575, Avenue Abdelkrim Khattabi, 40000, Guéliz-Marrakesh, Morocco

ARTICLE INFO

Article history:
Received 13 December 2012
Received in revised form 23 June 2013
Accepted 26 June 2013
Available online 1 July 2013

Keywords:
Fractional Ornstein–Uhlenbeck processes
Discrete-time observation
Least squares estimator
Kolmogorov distance
Central limit theorem
Malliavin calculus

ABSTRACT

Let $\theta > 0$. We consider a one-dimensional fractional Ornstein–Uhlenbeck process defined as $dX_t = -\theta X_t dt + dB_t$, $t \geq 0$, where B is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$. We are interested in the problem of estimating the unknown parameter θ. For that purpose, we dispose of a discretized trajectory, observed at n equidistant times $t_i = i\Delta_n$, $i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the 'observation window'. We assume that $\Delta_n \to 0$ and $T_n \to \infty$ as $n \to \infty$. As an estimator of θ we choose the least squares estimator (LSE) $\widehat{\theta}_n$. The consistency of this estimator is established. Explicit bounds for the Kolmogorov distance, in the case when $H \in (\frac{1}{2}, \frac{3}{4})$, in the central limit theorem for the LSE $\widehat{\theta}_n$ are obtained. These results hold without any kind of ergodicity on the process X.

1. Introduction

In this paper we consider a fractional Ornstein–Uhlenbeck process $X = (X_t, t \geq 0)$. That is, it solves the linear stochastic differential equation

$$X_0 = x_0; \qquad dX_t = -\theta X_t dt + dB_t, \quad t \geq 0, \tag{1}$$

where $x_0 \in \mathbb{R}$, $B = (B_t, t \geq 0)$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\theta > 0$ is an unknown parameter.

Assume that the process X is observed equidistantly in time with the step size Δ_n: $t_i = i\Delta_n$, $i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the 'observation window'. The purpose of this paper is to study the least squares estimator (LSE) $\widehat{\theta}_n$ of θ based on the sampling data X_{t_i}, $i = 0, \ldots, n$.

The LSE $\widehat{\theta}_n$ is obtained as follows: $\widehat{\theta}_n$ minimizes

$$\theta \mapsto \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} + \theta X_{t_{i-1}} \Delta_n \right|^2,$$

where $t_i = i\Delta_n$, $i = 0, \ldots, n$. Thus $\widehat{\theta}_n$ is given by

$$\widehat{\theta}_n = -\frac{\sum_{i=1}^{n} X_{t_{i-1}} (X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2}. \tag{2}$$

* Tel.: +212 670012297.
E-mail address: k.essebaiy@uca.ma.

0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.06.032

Also, by using (1), we arrive to the following formula:

$$\widehat{\theta}_n - \theta = -\frac{\sum_{i=1}^{n} X_{t_{i-1}} U_i}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2} \tag{3}$$

where

$$U_i = \theta \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_s) ds + B_{t_i} - B_{t_{i-1}}, \quad i = 1, \ldots, n.$$

The parametric estimation problems for fractional diffusion processes based on continuous-time observations have been studied e.g. in Kleptsyna and Le Breton (2002), Tudor and Viens (2007), Prakasa Rao (2005, 2010) via maximum likelihood method. Recently, the parametric estimation of continuously observed fractional Ornstein–Uhlenbeck process given in (1) is studied by using the least squares estimator (LSE) defined by

$$\widetilde{\theta}_T = -\frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt}.$$

In the case $\theta > 0$, Hu and Nualart (2010) proved that the LSE $\widetilde{\theta}_T$ of θ is strongly consistent and asymptotically normal. In addition, they also proved that the following estimator

$$\overline{\theta}_T = \left(\frac{1}{H\Gamma(H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}$$

is strongly consistent and asymptotically normal. In the case $\theta < 0$, Belfadli et al. (2011) established that the LSE $\widetilde{\theta}_T$ of θ is strongly consistent and asymptotically Cauchy.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for fractional diffusion processes based on discrete observations.

There exists a rich literature on the parameter estimation problem for diffusion processes driven by Brownian motions based on discrete observations, see Prakasa Rao (1988) and Prakasa Rao (2010) for more details about this point. For the fractional Ornstein–Uhlenbeck process (1), Hu and Song (2013), motivated by the estimator $\overline{\theta}_T$, proved that the following estimator

$$\underline{\theta}(n) = \left(\frac{1}{nH\Gamma(H)} \sum_{i=1}^{n} X_{t_i}^2 \right)^{-\frac{1}{2H}}$$

is strongly consistent and a Berry–Esséen type theorem for $\underline{\theta}(n)$ is obtained. In this paper, we focus our discussion on the LSE case.

In general, the study of the asymptotic distribution of any estimator is not very useful for practical purposes unless the rate of convergence of its distribution is known. The rate of convergence of the distribution of LSE for some diffusion processes driven by Brownian motions based on discrete time data was studied e.g. in Mishra and Prakasa Rao (2007). To the best of our knowledge there is no study of this problem for the distribution of the LSE of the unknown drift parameter in Eq. (1). Our goal in the present paper is to investigate the consistency and the rate of convergence to normality of the LSE $\widehat{\theta}_n$ defined in (2).

Recall that, if Y, Z are two real-valued random variables, then the Kolmogorov distance between the law of Y and the law of Z is given by

$$d_{\mathrm{Kol}}(Y, Z) = \sup_{-\infty < z < \infty} |P(Y \le z) - P(Z \le z)|.$$

Let us now describe the results we prove in this work. In Theorem 3.3 we show that the consistency of $\widehat{\theta}_n$ as $\Delta_n \to 0$ and $n\Delta_n \to \infty$ holds true if $H \in (\frac{1}{2}, 1)$. When $H \in (\frac{1}{2}, \frac{3}{4})$ we use the Malliavin calculus, the so-called Stein's method on Wiener chaos introduced by Nourdin and Peccati (2007) and the technical Lemmas 3.4 and 3.5 proved respectively by Michael and Pfanzagl (1971) and Babu et al. (1978), to derive Berry–Esséen-type bounds in the Kolmogorov distance for the LSE $\widehat{\theta}_n$ (Theorems 3.6 and 3.7).

We proceed as follows. In Section 2 we give the basic tools of Malliavin calculus for the fractional Brownian motion needed throughout the paper. Section 3 contains our main results, concerning the consistency and the rate of convergence of $\widehat{\theta}_n$.

2. Preliminaries

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more complete presentation on the subject, see Nualart (2006) and Alòs and Nualart (2003).