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A note on the stochastic properties of a scale change random effects model[☆]

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ABSTRACT

The scale change model in survival analysis incorporates unobserved heterogeneity through a frailty that enters the baseline hazard function to change the time scale. In this paper we examine the stochastic properties of the mixtures of scale change model and build dependence between the overall population variable and the frailty variable. We also carry out stochastic comparisons between overall population variables when their respective frailty or baseline variables are ordered in the sense of various stochastic orders. Finally, we demonstrate how the variation of the baseline variable has an effect on the model.

1. Introduction

Many populations encountered in survival analysis are often not homogeneous. Individuals differ in their susceptibility to causes of death, response to treatment and influence of various risk factors. Ignoring this heterogeneity can result in misleading conclusions. To deal with these problems, the proportional hazard frailty model was introduced. The frailty variable Z represents genetic and environmental influence, risk factors and susceptibilities for each individual. Given that $Z = z$, the hazard rate for an individual takes the form

$$h(t|z) = zh_0(t)$$

under proportional hazard frailty model, where $h_0(t)$ is a baseline hazard rate function, the same for all individuals (cf. Vaupel et al. (1979)).

As an alternative to above model, the scale change model has been developed in survival analysis (Louis, 1981; Tsiatis, 1990; Anderson and Louis, 1995). This model incorporates a frailty into baseline hazard function to change the time scale.

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For this model, given that $Z = z$, the survival time T for an individual has a hazard rate function as

$$h(t|z) = zh_0(zt), \tag{1.1}$$

where $h_0(t)$ is a baseline hazard rate function. Another description of this model is

$$\overline{F}(t|z) = \overline{G}(zt), \tag{1.2}$$

where $\overline{F}(t|z)$ is the survival function of the conditional random variable $[T|Z = z]$, and

$$\overline{G}(t) = \exp\left\{-\int_0^t h_0(x)\mathrm{d}x\right\}$$

is the survival function of baseline random variable denoted as Y. The frailty variable is basically considered unobserved and hence the individual level model (1.1) is unobservable. Therefore, it would be reasonable to consider the corresponding population level. The population survival function, based on (1.2), can be written as

$$\overline{F}(t) = \mathsf{E}[\overline{G}(Zt)]. \tag{1.3}$$

The overall population hazard rate function $h(t)$ is then associated with the baseline hazard rate function through the equation

$$h(t) = \mathsf{E}[Zh_0(Zt)|T > t],$$

(see Anderson and Louis (1995)).

The present paper focuses on the mixture model in (1.3). Section 2 derives negative dependence property between the frailty variable and the overall population variable and presents a preservation of DLR class of aging distributions. In Section 3, we stochastically compare population variables arising from different choices of frailty distributions or baseline distributions. We finally demonstrate how the variation of the baseline variable has an effect on the model in Section 4.

Throughout this paper, all random variables under consideration are nonnegative, "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively. All expectations and integrals are assumed to exist whenever they appear.

For the sake of completeness, let us recall some stochastic orders and aging notions which will be used in the sequel.

Definition 1.1. Let X_1 and X_2 be two absolutely continuous random variables with density functions f_1 and f_2, distribution functions F_1 and F_2, survival functions $\overline{F}_1$ and $\overline{F}_2$. X_1 is said to be smaller than X_2 in the

(i) *likelihood ratio order* (denoted by $X_1 \le_{\mathrm{lr}} X_2$) if $f_2(x)/f_1(x)$ is increasing in x;
(ii) *hazard rate order* (denoted by $X_1 \le_{\mathrm{hr}} X_2$) if $\overline{F}_2(x)/\overline{F}_1(x)$ is increasing in x;
(iii) *reversed hazard rate order* (denoted by $X_1 \le_{\mathrm{rh}} X_2$) if $F_2(x)/F_1(x)$ is increasing in x;
(iv) *mean residual order* (denoted by $X_1 \le_{\mathrm{mrl}} X_2$) if $\mathsf{E}[X_1 - x|X_1 \ge x] \le \mathsf{E}[X_2 - x|X_2 \ge x]$ for all $x \ge 0$;
(v) *mean inactivity time order* (denoted by $X_1 \le_{\mathrm{mit}} X_2$) if $\mathsf{E}[x - X_1|X_1 \le x] \le \mathsf{E}[x - X_2|X_2 \le x]$ for all $x \ge 0$.

The relationship among above stochastic orders is shown in the following chain (see Shaked and Shanthikumar (2007) and Müller and Stoyan (2002)):

$$X_1 \le_{\mathrm{lr}} X_2 \Longrightarrow X_1 \le_{\mathrm{hr}}[\le_{\mathrm{rh}}]X_2 \Longrightarrow X_1 \le_{\mathrm{mrl}}[\le_{\mathrm{mit}}]X_2.$$

Definition 1.2 (*Lai and Xie (2006) and Barlow and Proschan (1981)*). Let X be a random variable with distribution function F and survival function $\overline{F}$. X is said to be

(a) *decreasing likelihood ratio* (denoted by DLR) if $\log f(x)$ is convex in $x \ge 0$, where $f(x)$ is the density function of X;
(b) *decreasing failure rate* (denoted by DFR) if $\log \overline{F}(x)$ is convex in $x \ge 0$.

2. Negative dependence and aging properties

Dependence is of great interest in survival analysis, reliability theory, actuarial science, and many other areas closely related to probability and statistics. It can be seen from (1.3) that the overall population variable T is linked to the baseline variable Y through the frailty variable Z. Therefore, the examination of the dependence property between T and Z can help one have a better understanding of the mechanism of this model. For the sake of convenience, we denote by g $[G, \overline{G}]$, h $[H, \overline{H}]$ and f $[F, \overline{F}]$ the density (distribution, survival) functions of Y, Z and T, respectively.

Let us recall the strongest negative dependent concept of two random variables, called *negatively likelihood ratio dependent* (see Nelsen (1999)). T and Z are negatively likelihood ratio dependent if their joint density function $f(t, z)$ satisfies

$$f(t_1, z_1)f(t_2, z_2) - f(t_1, z_2)f(t_2, z_1) \le 0$$

for all $t_2 \ge t_1 \ge 0$ and $z_2 \ge z_1 \ge 0$. Recall that a function $h(x, y) \ge 0$ is TP_2 $[\mathrm{RR}_2]$ (cf. Karlin (1968, p. 107)) if $h(x_1, y_1)h(x_2, y_2) \ge [\le]h(x_1, y_2)h(x_2, y_1)$ for all $x_1 \le x_2$ and $y_1 \le y_2$.