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On convergence of random walks generated by compound Cox processes to Lévy processes

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ABSTRACT

A functional limit theorem is proved establishing weak convergence of random walks generated by compound doubly stochastic Poisson processes to Lévy processes in the Skorokhod space. As corollaries, theorems are proved on convergence of random walks with jumps having finite variances to Lévy processes with mixed normal distributions, and in particular, to stable Lévy processes.

1. Introduction

In financial mathematics the evolution of (the logarithms of) stock prices and financial indexes on small time horizons is often modeled by random walks. The simplest example of such an approach is the Cox–Ross–Rubinstein model (see, e.g., Shiryaev (1999)). At the same time most successful (adequate) models of the dynamics of (the logarithms of) financial indexes on large time horizons are subordinated Wiener processes such as generalized hyperbolic processes and, in particular, variance-gamma (VG) processes and normal\\inverse Gaussian (NIG) processes; see Shiryaev (1999). Functional limit theorems are a quite natural link between random walks and subordinated Wiener processes. The operation of subordination gives a good explanation of the presence of heavy tails in the empirical distributions of the increments of (the logarithms of) stock prices and financial indexes.

Stable Lévy processes were among the first heavy-tailed models to be successfully applied in practice. According to the approach based on classical limit theorems of probability theory, non-normal stable Lévy processes can appear as limits in functional limit theorems for random walks only if the variances of elementary jumps are infinite. In turn, this is possible

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only if the probabilities of arbitrarily large jumps are positive. Unfortunately, the latter condition looks very doubtful from the practical viewpoint. Therefore, within the framework of the classical approach, the theoretical explanation of stable Lévy processes as adequate models for the evolution of stock prices or financial indexes is at least questionable.

In Korolev (1997, 1998) it was shown that stable laws can appear as limit distributions for sums of independent identically distributed random variables with *finite* variances, if the number of summands in the sum is random and its distribution belongs to the domain of attraction of a stable law concentrated on the nonnegative half-line. When random walks are considered, the scheme of random summation is a natural analog of the scheme of subordination of more general random processes. In Gnedenko and Korolev (1996) and Korolev (2000) the proposal was made of modeling the evolution of non-homogeneous chaotic stochastic processes, in particular, of the dynamics of stock prices and financial indexes, by random walks generated by compound doubly stochastic Poisson processes (compound Cox processes). Special continuous-time random walks were considered in Granovsky and Zeifman (1997) and Zeifman (1995). A *doubly stochastic Poisson process* (also called a *Cox process*) is a stochastic point process of the form $N_1(\Lambda(t))$, where $N_1(t)$, $t \geq 0$, is a homogeneous Poisson process with unit intensity and the stochastic process $\Lambda(t)$, $t \geq 0$, is independent of $N_1(t)$ and possesses the following properties: $\Lambda(0) = 0$, $\mathsf{P}(\Lambda(t) < \infty) = 1$ for any $t > 0$, and the sample paths of $\Lambda(t)$ do not decrease and are right-continuous.

In Kashcheev (2001) some functional limit theorems were proved for compound Cox processes with square integrable leading processes $\Lambda(t)$. However, the class of limit processes for compound Cox processes with such leading processes and jumps having finite variances is not wide enough. In particular, it cannot contain any stable Lévy processes except the Wiener process.

The aim of the present paper is to fill this gap and generalize the results mentioned above. Section 2 deals with basic definitions and auxiliary results. The main part of the paper is in Section 3. Here functional limit theorems will be proved describing the convergence of compound Cox processes with jumps possessing finite variances to Lévy processes from a rather wide class containing, in particular, stable Lévy processes.

2. Basic definitions and auxiliary results

Let $D = D[0, 1]$ be the space of real functions defined on $[0, 1]$, that are right-continuous and have finite left-side limits.

Let $\mathcal{F}$ be the class of strictly increasing mappings of $[0, 1]$ onto itself. Let f be a non-decreasing function on $[0, 1]$ with $f(0) = 0, f(1) = 1$. Set $\|f\| = \sup_{s\neq t} |\log[(f(t) - f(s))/(t - s)]|$. If $\|f\| < \infty$, then the function f is continuous and strictly increasing and, hence, belongs to $\mathcal{F}$.

Define the distance $d_0(x, y)$ in the set $D[0, 1]$ as the greatest lower bound of the set of positive numbers ϵ for which $\mathcal{F}$ contains a function f such that $\|f\| \leq \epsilon$ and $\sup_t |x(t) - y(f(t))| \leq \epsilon$.

It can be shown that the space $D[0, 1]$ is complete with respect to the distance d_0. The metric space $\mathcal{D} = (D[0, 1], d_0)$ is called the *the Skorokhod space*. Everywhere in what follows we will consider stochastic processes as $\mathcal{D}$-valued random elements.

Let $X, X_1, X_2, \ldots$ be $\mathcal{D}$-valued random elements. Let T_X be a subset of $[0, 1]$ such that $0 \in T_X$, $1 \in T_X$ and if $0 < t < 1$, then $t \in T_X$ if and only if $\mathsf{P}(X(t) \neq X(t-)) = 0$. The following theorem establishing sufficient conditions for the weak convergence of stochastic processes in $\mathcal{D}$ (denoted below as $\Longrightarrow$ and assumed as $n \to \infty$) is well-known.

Theorem A. *Let* $(X_n(t_1), \ldots, X_n(t_k)) \Longrightarrow (X(t_1), \ldots, X(t_k))$ *for any natural k and $t_1, \ldots, t_k$ belonging to T_X. Let* $\mathsf{P}(X(1) \neq X(1-)) = 0$ *and let there exist a non-decreasing continuous function F on $[0, 1]$ such that for any $\epsilon > 0$,*

$$\mathsf{P}(|X_n(t) - X_n(t_1)| \geq \epsilon, |X_n(t_2) - X_n(t)| \geq \epsilon) \leq \epsilon^{-2\nu}[F(t_2) - F(t_1)]^{2\gamma} \quad (1)$$

for $t_1 \leq t \leq t_2$ and $n \geq 1$, where $\nu \geq 0$, $\gamma > 1/2$. Then $X_n \Longrightarrow X$.

The proof of Theorem A can be found in, for example, Billingsley (1968).

Everywhere in what follows the symbol $\stackrel{d}{=}$ stands for the coincidence of distributions.

By a Lévy process we will mean a stochastic process $X(t)$, $t \geq 0$, possessing the following properties: (i) $X(0) = 0$ almost surely; (ii) $X(t)$ is a process with independent increments, that is, for any $N \geq 1$ and $t_0, t_1, \ldots, t_N$ $(0 \leq t_0 \leq t_1 \leq \cdots \leq t_N)$ the random variables $X(t_0), X(t_1) - X(t_0), \ldots, X(t_N) - X(t_{N-1})$ are jointly independent; (iii) $X(t)$ is a homogeneous process, that is, $X(t+h) - X(t) \stackrel{d}{=} X(s+h) - X(s)$ for any $s, t, h > 0$; (iv) the process $X(t)$ is stochastically continuous, that is, for any $t \geq 0$ and $\epsilon > 0$, $\lim_{s\to t} \mathsf{P}(|X(t) - X(s)| > \epsilon) = 0$; (v) sample paths of the process $X(t)$ are right-continuous and have finite left-side limits.

Denote the characteristic function of the random variable $X(t)$ as $\psi_t(s)$ ($\psi_t(s) = \mathsf{E}e^{isX(t)}$, $s \in \mathbb{R}$). The following statement describes a well-known property of Lévy processes.

Lemma 1. *Let $X = X(t)$, $t \geq 0$, be a Lévy process. For any $t > 0$ the characteristic function of the random variable $X(t)$ is infinitely divisible and has the form*

$$\psi_t(s) = [\psi_1(s)]^t = \left[\mathsf{E}\, e^{isX(1)}\right]^t, \quad s \in \mathbb{R}. \quad (2)$$