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Remarks on the speed of convergence of mixing coefficients and applications

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ABSTRACT

We study dependence coefficients for copula-based Markov chains. We provide new tools to check the convergence rates of mixing coefficients of copula-based Markov chains. We apply results to the Metropolis–Hastings algorithm. A necessary condition for symmetric copulas is given and mixtures of copulas are studied.

1. Introduction

This work is motivated by questions raised after reading Chen et al. (2009) and Beare (2010). Chen et al. (2009) have shown that Markov chains generated by the Clayton, Gumbel or Student copulas are geometrically ergodic. They used in their paper quantile transformations and small sets to show geometric ergodicity, but could not handle for instance the mixture of these copulas. In a recent paper, Longla and Peligrad (2012) have shown that these examples are actually exponential ρ-mixing. We have also answered the open question on geometric ergodicity of convex combinations of geometrically ergodic reversible Markov chains.

Quantifying the dependence among two or more random variables has been an enduring task for statisticians. Copulas are full measures of dependence among components of random vectors. Unlike marginal and joint distributions, which are directly observable, a copula is a hidden dependence structure that couples a joint distribution with its marginals. An early statistical application of copulas was given by Clayton (1978), where the dependence between two survival times in a multiple events study is modeled by the so-called Clayton copula

$$C(x, y) = (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha}, \quad \alpha \geq 0.$$

The literature on copulas is growing fast. An excellent overview, guide to the literature and applications, is due to Embrechts et al. (2003). In later research on copulas, a driving force has been in financial risk management, where they are used to model dependence among different assets in a portfolio. Nelsen's monograph (2006) can be regarded as one of the best books for an introduction to copulas.

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1.1. Definitions

1.1.1. 2-copulas

A 2-copula is a function $C : [0,1] \times [0,1] \to [0,1] = I$ for which $C(0,y) = C(x,0) = 0$ (C is grounded), $C(1,x) = C(x,1) = x$ "each coordinate is uniform on I", and for all $[x_1,x_2]\times[y_1,y_2] \subset I^2$, $C(x_1,y_1)+C(x_2,y_2)-C(x_1,y_2)-C(x_2,y_1) \geq 0$. The definition does not mention probability, but in fact these conditions imply that C is the joint cumulative distribution function of two random variables with marginal distributions uniform on I. It follows from the definition (see Chapter 1 in Nelsen, 2006) that the function C is non-decreasing in each of its variables and has partial derivatives almost everywhere with values between 0 and 1. The partial derivative of $C(x,y)$ with respect to x is a non-decreasing function of y and similarly with respect to y. A convex combination of 2-copulas is a 2-copula.

If X_1, X_2 are random variables with joint distribution F and marginal distributions F_1, F_2, then the function $C(x,y)$ defined by $C(F_1(x_1), F_2(x_2)) = F(x_1,x_2)$ is a copula. Moreover, if the random variables are continuous, then the copula is uniquely defined by the joint distribution and the marginal distributions by the formula $F(F_1^{-1}(x_1), F_2^{-1}(x_2)) = C(x_1,x_2)$. This fact is known as Sklar's theorem. The implication of the Sklar's Theorem is that, after standardizing the effects of marginals, the dependence among components of $X = (X_1,X_2)$ is fully described by the copula. Indeed, most conventional measures of dependence can be explicitly expressed in terms of the copula. We will use in this paper the following conventional notation: $\|g\|_2^2 = \int_I g^2(x)dx$, for $i = 1,2$, $A_{,i}(x_1,x_2) = \frac{\partial A(x_1,x_2)}{\partial x_i}$, $c(x,y)$ will be used for the density of $C(x,y)$, and $\mathcal{R}$ will be used for the Borel σ-algebra of I.

1.1.2. Copulas and Markov processes

Copulas have been shown to be a more flexible way to define a Markov process, as in the case when one suspects that the marginal distributions of the states are not related to the distribution of the initial state. Using copulas will allow changes in single marginal distributions, without having to change all other distributions in the chain.

A stationary Markov chain $(X_n, n \in \mathbb{Z})$ can be defined by a copula $(C(x,y))$ and a one dimensional marginal distribution. For stationary Markov chains with uniform marginals on $[0,1]$, the transition probabilities for all $n \in \mathbb{Z}$ are $P(X_n \in A|X_{n-1} = x) = C_{,1}(x,y)$ for sets $A = (-\infty, y]$ (for more details, see Theorem 3.1 in Darsow et al., 1992). This relationship was used by Chen et al. (2009) to show that stationary Markov processes defined by the Clayton, Gumbel or Student copulas are geometrically ergodic.

1.1.3. Dependence coefficients

Many dependence coefficients have been studied in the literature, such as $\alpha_n, \beta_n, \rho_n, \phi_n$ among others. In this paper, we will mainly use the last 3 coefficients defined as follows.

Given σ-fields $\mathscr{A}, \mathscr{B}$:

$$\beta(\mathscr{A},\mathscr{B}) = \mathbb{E}\sup_{B\in\mathscr{B}} |P(B|\mathscr{A}) - P(B)|$$

$$\rho(\mathscr{A},\mathscr{B}) = \sup_{f\in L^2(\mathscr{A}),g\in L^2(\mathscr{B})} corr(f,g),$$

$$\phi(\mathscr{A},\mathscr{B}) = \sup_{B\in\mathscr{B},A\in\mathscr{A},P(A)>0} |P(B|A) - P(B)|.$$

Given the alternative form of the transition probabilities for a Markov chain generated by a copula and a marginal distribution with strictly positive density, it was shown in Longla and Peligrad (2012) that these coefficients have the following simple form when the copula for (X_0, X_n) is absolutely continuous with density c_n, $\mathscr{A} = \sigma(X_i, i \leq 0)$ and $\mathscr{B} = \sigma(X_i, i \geq n)$:

$$\beta_n = \int_0^1 \sup_{B\in\mathcal{R}} \left| \int_B (c_n(x,y) - 1)dy \right| dx,$$

$$\phi_n = \sup_{B\in\mathcal{R}} \operatorname{ess\,sup}_x \left| \int_B (c_n(x,y) - 1)dy \right|,$$

$$\rho_n = \sup \left\{ \int_0^1 \int_0^1 c_n(x,y)f(x)g(y)dxdy : \|g\|_2 = \|f\|_2 = 1, \mathbb{E}(f) = \mathbb{E}(g) = 0 \right\}.$$

In general the following inequalities hold (for more, see Theorems 7.4 and 7.5 in Bradley, 2007):

$$\beta_n \leq \phi_n, \qquad \rho_n \leq 2\sqrt{\phi_n}, \qquad \rho_n \leq (\rho_1)^n. \tag{1}$$

These coefficients are defined to assess the dependence structure of the Markov process and provide necessary conditions for CLT and functional CLT and their rates of convergence. Some examples can be found in Peligrad (1997, 1993) and the references therein. A stochastic process is said to be α-mixing if $\alpha_n \to 0$; β-mixing if $\beta_n \to 0$ or ρ-mixing if $\rho_n \to 0$. The process is exponentially mixing if the convergence rate is exponential. A stochastic process is said to be geometrically