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Characterizations of discrete distributions using reliability concepts in reversed time[☆]

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ABSTRACT

In the present work we establish characterizations of some discrete distributions using properties of the reversed hazard rate and reversed mean residual life. Discrete distributions having a constant reversed hazard rate, the reversed lack of memory property, and the product of the reversed hazard rate and the mean residual life a constant are identified. Some properties of the reversed variance residual life and a characterization are also discussed.

1. Introduction

There has been growing interest in recent times in the study of reliability functions in reversed time and their applications. The functions of primary interest discussed in the continuous case in the literature are the reversed hazard rate (Block et al., 1998; Finkelstein, 2002), the reversed mean (variance) residual life or mean (variance) inactivity time (Nanda et al., 2003; Li and Lu, 2003; Kundu and Nanda, 2010), and the reversed percentile residual life (Nair and Vineshkumar, 2010). One can also find applications of these concepts in different topics of investigation in the above papers and their references. Some analogous results and characterizations in the case of the reversed hazard rate and mean residual life for discrete distributions are discussed in Gupta et al. (2006) and Goliforushani and Asadi (2008).

It is known (Block et al., 1998) that in the case of absolutely continuous distributions on the positive real line there does not exist a model with constant reversed hazard rate. The main objective of the present article is to show that this and several other properties not enjoyed by absolutely continuous distributions can hold good in the discrete case. Specifically, we identify discrete models for which the reversed hazard rate is constant, the product of the reversed hazard rate and the reversed mean residual life is constant, etc. Some properties of the reversed variance residual life and a characterization are also presented.

2. Characterizations

Let X be a discrete random variable taking values in the set $S = (0, 1, 2, \dots, b)$, where b is a positive integer and can be ∞. We denote the probability mass function and distribution function of X by $f(x)$ and $F(x)$ respectively. Then the reversed

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hazard rate of X is defined as

$$\lambda(x) = P(X = x|X \le x) = \frac{f(x)}{F(x)}. \tag{2.1}$$

The distribution of X is determined uniquely by $\lambda(x)$ through the formula

$$F(x) = \prod_{t=x+1}^{b} (1 - \lambda(t)). \tag{2.2}$$

Further, the random variable X is said to satisfy the reversed lack of memory property if and only if

$$P(X \le t|X \le t+s) = P(X \le 0|X \le s) \tag{2.3}$$

for all t, s in S. The property (2.3) can be interpreted in the following way in the context of maintenance problems. When X represents the lifetime of a device, its inactivity time (time since failure) is independent of the age of the device.

With these definitions, we have the following characterizations, in which we assume that $b < \infty$.

Theorem 2.1. *The random variable X is distributed as*

$$F(x) = \begin{cases} (1+c)^{x-b}, & x = 0, 1, 2, \ldots, b;\ c > 0,\ b < \infty \\ 1, & x \ge b \end{cases} \tag{2.4}$$

if and only if any one of the following conditions is satisfied:
(a) $\lambda(x) = k$*, a constant for all* $x > 0$*, where* $0 < k < 1$*;*
(b) *X has the reversed lack of memory property.*

Proof. From (2.4),

$$f(x) = \begin{cases} (1+c)^{-b}, & x = 0 \\ c(1+c)^{x-b-1} & x = 1, 2, \ldots, b, \end{cases}$$

and accordingly

$$\lambda(x+1) = \frac{c}{1+c}, \quad x = 0, 1, 2, \ldots, b-1.$$

Notice that $\lambda(0) = 1$ for all discrete distributions with zero as the left end point for their supports. The converse of (a) follows from Eq. (2.2) on taking $k = \frac{c}{1+c}$. That the distribution (2.4) satisfies (b) can be easily verified from (2.3). On the other hand when the reversed lack of memory property holds, we have

$$F(t+s)F(0) = F(t)F(s)$$

or

$$a(t+s) = a(t)a(s), \quad \text{where } a(t) = \frac{F(t)}{F(0)}.$$

To solve the functional equation, set $s = 1$, so we have

$$a(t+1) = a(t)a(1) = \cdots = [a(1)]^{t+1}$$

for all $t = 1, 2, \ldots, b-1$. Equivalently

$$F(x+1) = \frac{[F(1)]^{x+1}}{[F(0)]^{x}} \tag{2.5}$$

with

$$F(b) = \frac{p^b}{[F(0)]^{b-1}} = 1$$

or

$$F(0) = p^{\frac{b}{b-1}}, \quad p = F(1).$$

Substituting for $F(0)$ in (2.5) and setting $p = (1+c)^{-1}$, we have (2.4), and the proof is completed. □

Remark 1. The terms in the probability mass function for $x = 1, 2, 3, \ldots$ are in increasing geometric progression with common ratio $(1+c)$, as opposed to the usual geometric distribution where the terms are decreasing. We shall refer to (2.4) as the reversed geometric distribution in view of the reversal of the monotonicity of successive probabilities. Thus $\lambda(x)$ is constant $\Leftrightarrow$ X has a reversed geometric law $\Leftrightarrow$ X has the reverse lack of memory property.

The reversed mean residual lifetime is defined as

$$r(x) = E(x - X|X < x) = \frac{1}{F(x-1)} \sum_{t=1}^{x} F(t+1) \tag{2.6}$$